Exam 1 Review

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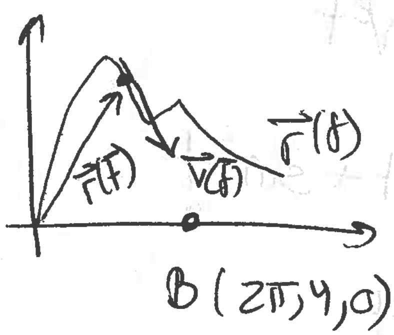
Ex1) suppose $(\vec{A}+\vec{B})$ and $(\vec{A}-\vec{B})$ are orthogonal. What can you conclude about $\vec{A}$ and $\vec{B}$?

$$\Rightarrow 0 = (\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) = \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} - \vec{B}\cdot\vec{B}$$

$$= \|\vec{A}\|^2 - \|\vec{B}\|^2 \Rightarrow \|\vec{A}\| = \|\vec{B}\|$$

Ex2) Person A is on an amusement park ride. He follows path $\vec{r}(t) = (t-\sin t)\hat{i} + (1-\cos(t))\hat{j} + 0\hat{k}$ for $0 \le t \le 2\pi$. Person B stands at $(2\pi, 4, 0)$. Person A stares straight ahead.

(a) when on the ride does person A stare straight at B?



we need $B-\vec{r}(t)$ to be parallel to $\vec{v}(t)$.

At this point, A stares at B.

$$\Rightarrow (B-\vec{r}(t)) \times \vec{v}(t) = 0 \quad \Rightarrow \|(B-\vec{r}(t)) \times \vec{v}(t)\| = 0$$

$$B-\vec{r}(t) = \langle 2\pi - t + \sin t,\ 4-1+\cos t,\ 0\rangle$$

$$= \langle 2\pi - t + \sin t,\ 3+\cos t,\ 0\rangle$$

$$\vec{r}(t) = \langle t-\sin t,\ 1-\cos t,\ 0\rangle$$

$$\vec{v}(t) = \langle 1-\cos t,\ \sin t,\ 0\rangle$$

$$\left(\vec{B}-\vec{r}(t)\right)\times\vec{v}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (2\pi-t+\sin t) & 3+\cos t & 0 \\ 1-\cos t & \sin t & 0 \end{vmatrix}$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}\begin{vmatrix} (2\pi-t+\sin t) & 3+\cos t \\ 1-\cos t & \sin t \end{vmatrix}$$

$$= \hat{k}\left[(2\pi-t+\sin t)\cdot\sin t - (3+\cos t)(1-\cos t)\right]$$

$$\|\vec{B}-\vec{r}(t)\times\vec{v}(t)\| = 0$$

$$\Rightarrow \cancel{2\pi\sin(t) - t\sin(t) + \sin^2(t) - 3 + 3\cos(t) + \cos(t) + \cos^2(t) = \ldots}$$

$$\Rightarrow \cancel{3 - 3\cos(t) - \cos^2 t = (2\pi-t)\sin t + \sin^2 t}$$

$$3 - 3\cos t + \cos t - \cos^2 t = (2\pi-t)\sin t + \sin^2 t$$

$$3 - 2\cos t = (2\pi-t)\sin t + \cos^2 t + \sin^2 t$$

$$3 - 2\cos t = (2\pi-t)\sin t + 1 \Rightarrow 2 - 2\cos(t) = (2\pi-t)\sin(t)$$

$$\Rightarrow 1-\cos(t) = \left(\pi - \frac{t}{2}\right)\sin t \Rightarrow \text{solve for } t.$$

(b) what is the arclength? (up to time u) time 0

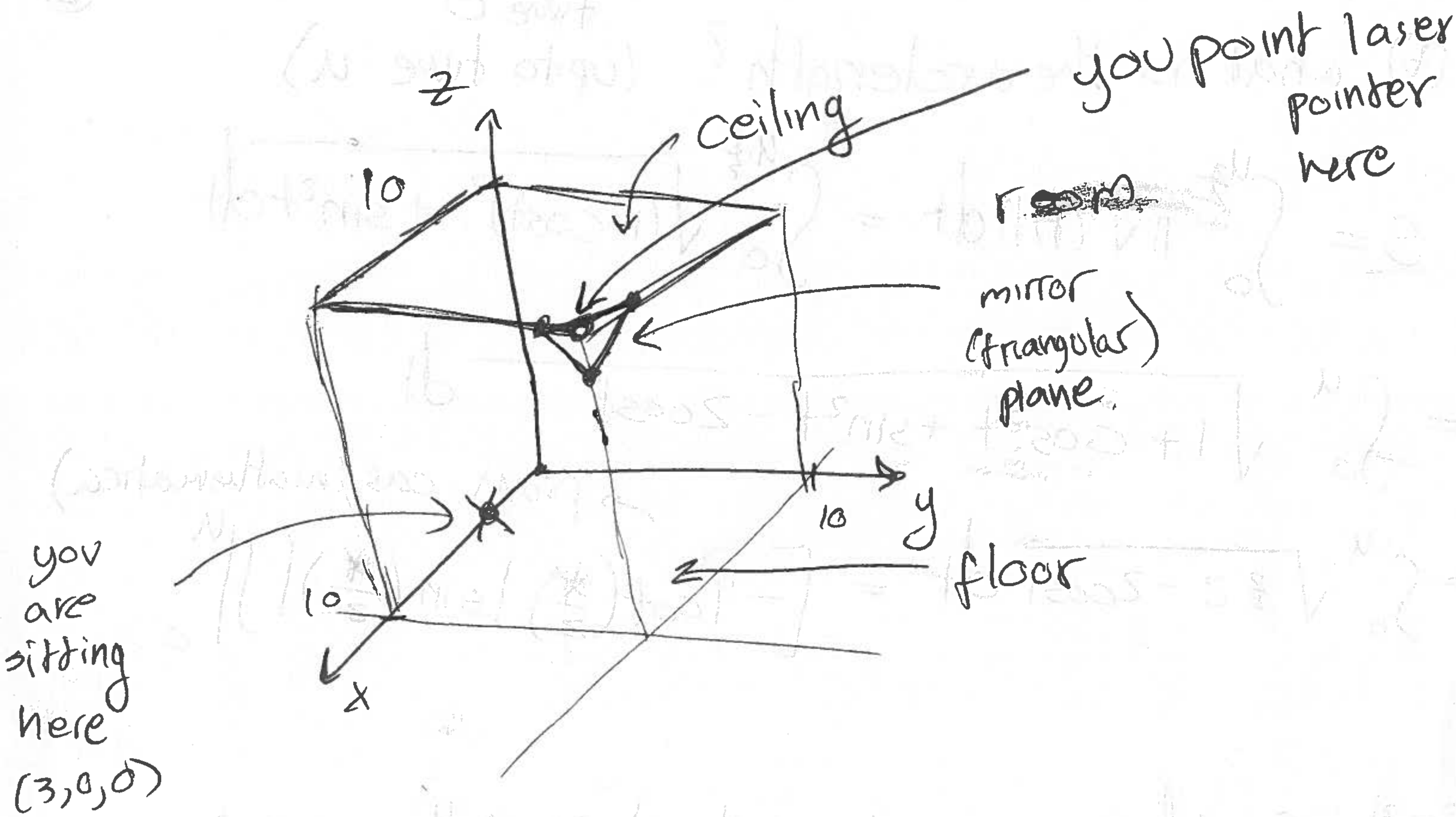
$$S = \int_0^{u} \|\vec{v}'(t)\| \, dt = \int_0^{u} \sqrt{(1-\cos t)^2 + \sin^2 t} \, dt$$

$$= \int_0^{u} \sqrt{1 + \cos^2 t + \sin^2 t - 2\cos t} \, dt$$

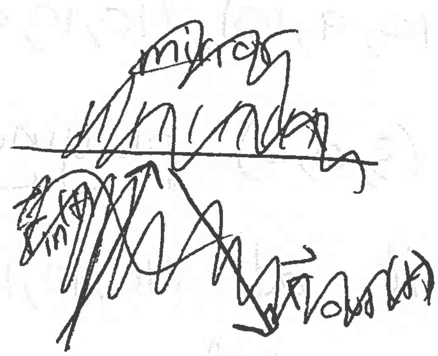
from cas (mathematica)

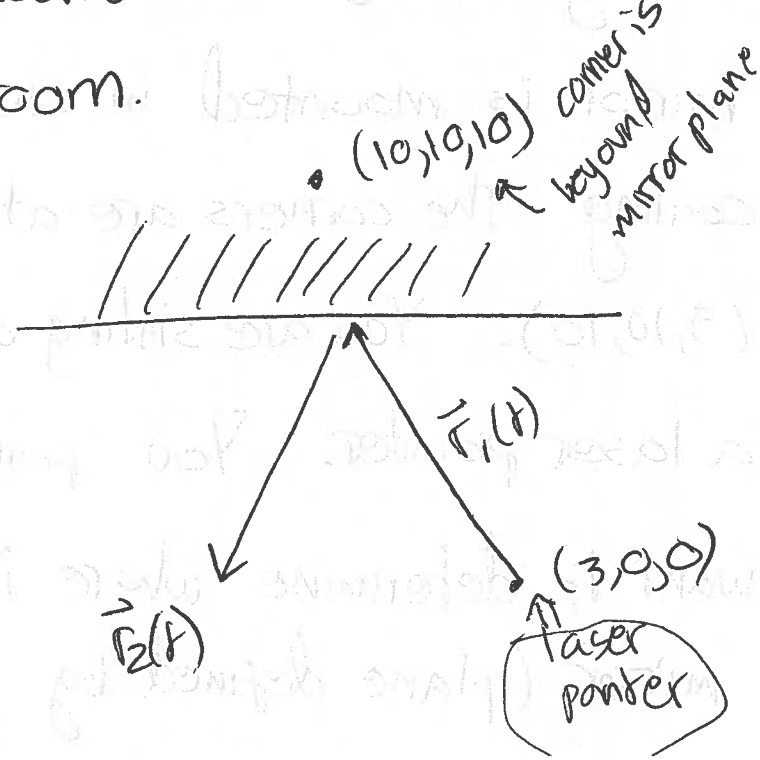
$$= \int_0^{u} \sqrt{2 - 2\cos t} \, dt = \left[-4\cot\left(\frac{t}{2}\right)\left|\sin\left(\frac{t}{2}\right)\right|\right]\Big|_0^{u}$$

Ex 3 Consider a room 10 units long in the x, y, z directions. The walls of the room are four planes: $x=0$, $x=10$, $y=0$, and $y=10$ and the floor and ceiling are $z=0$ and $z=10$. A flat triangular mirror is mounted in one of the corners of the ceiling. The corners are at $(10,9,10)$, $(10,10,9)$, and $(9,10,10)$. You are sitting at $(3,0,0)$ playing with a laser pointer. You point it at $(10,10,10)$ and want to determine where it's reflected from flat mirror (plane defined by 3 pts).



(a) If you aim your laser pointer directly at the corner of the room with coordinates (10,10,10), determine the coordinates where the reflected beam will hit the walls, or floor, of the room.



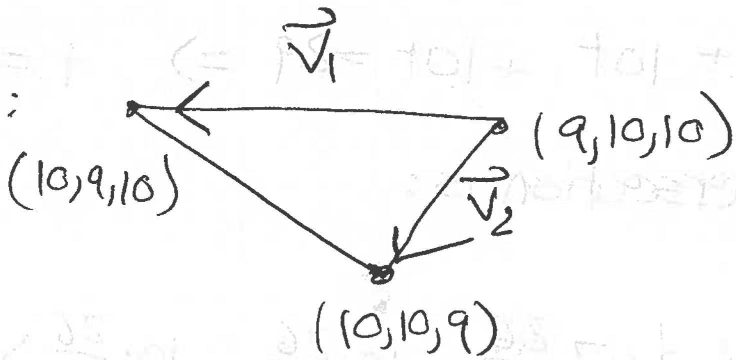


you are at $\overset{P}{(3,0,0)}$ aiming at $\overset{Q}{(10,10,10)}$

$$\vec{PQ} = \langle 7,10,10 \rangle$$

$$\vec{r_1}(t) = \langle 3,0,0 \rangle + t\langle 7,10,10 \rangle$$

mirror-plane :



3 points make a plane...

$\vec{n} = \vec{v_1} \times \vec{v_2}$ where $\vec{v_1} = \langle 1,-1,0 \rangle$ and $\vec{v_2} = \langle 1,0,-1 \rangle$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1,1,1 \rangle$$

take plane through point $(9,10,10)$:

$$1\cdot(x-9) + 1\cdot(y-10) + 1\cdot(z-10) = 0$$

$$\Rightarrow x+y+z = 29$$

We want to find where on the plane, the beam hits (the poin of intersection, of $r_1(t)$ and plane).

$$\vec{r_1}(t) = \langle 3+7t, 10t, 10t \rangle$$

so intersection of

$$(x-9)+(y-10)+(z-10)=0 \Rightarrow x+y+z=29$$

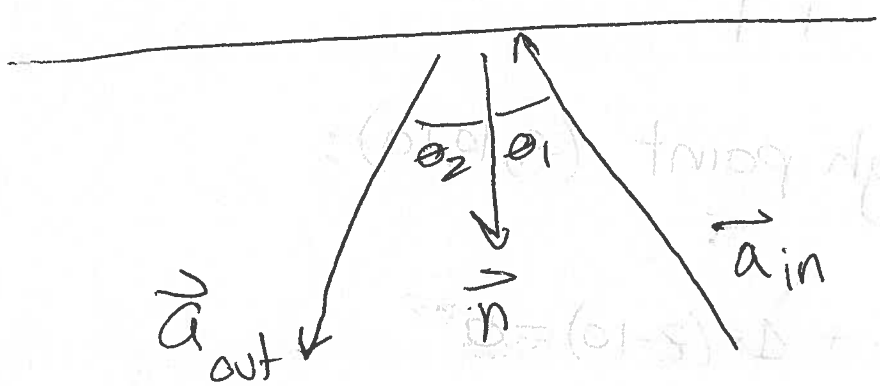
and $\vec{r}_1(t) = \langle 3+7t, 10t, 10t \rangle$

$$\Rightarrow (3+7t)+10t+10t=29 \Rightarrow t=\frac{26}{27}$$

point of intersection is:

$$\vec{r}_1\left(\frac{26}{27}\right) = \left\langle 3+7\frac{26}{27}, 10\cdot\frac{26}{27}, 10\cdot\frac{26}{27} \right\rangle = \left\langle \frac{263}{27}, \frac{260}{27}, \frac{260}{27} \right\rangle$$

this is where laser pointer beam intersects the mirror plan



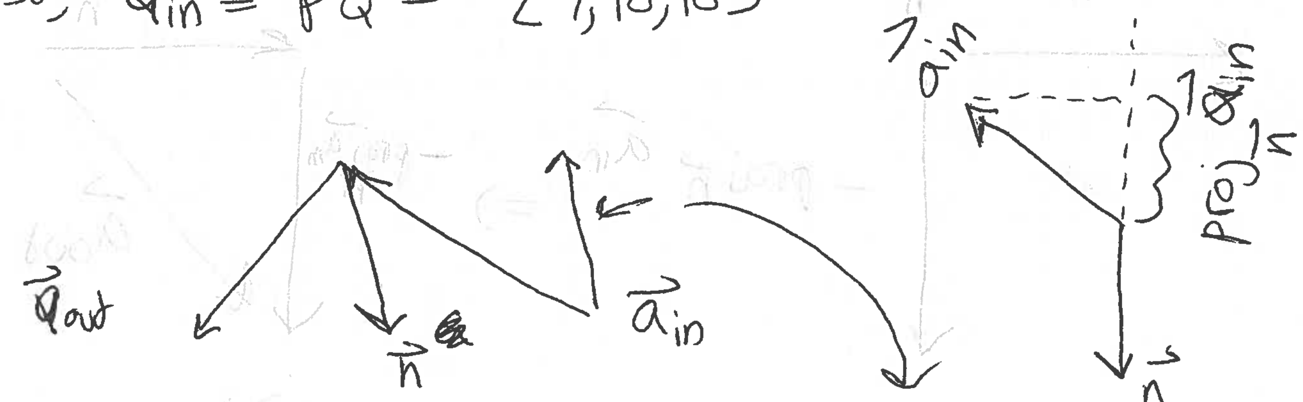
Hint given: an incoming ray of light and the surface normal where the ray hits the surface, form a plane. The reflected ray is in the same plane. Also $\theta_1 = \theta_2$ (angle btwn reflected plane and normal and btwn normal and reflected ray).

want to find vector $\vec{a}_{out}$ and use it to construct $\vec{r}_2(t)$ given point of intersection we computed.

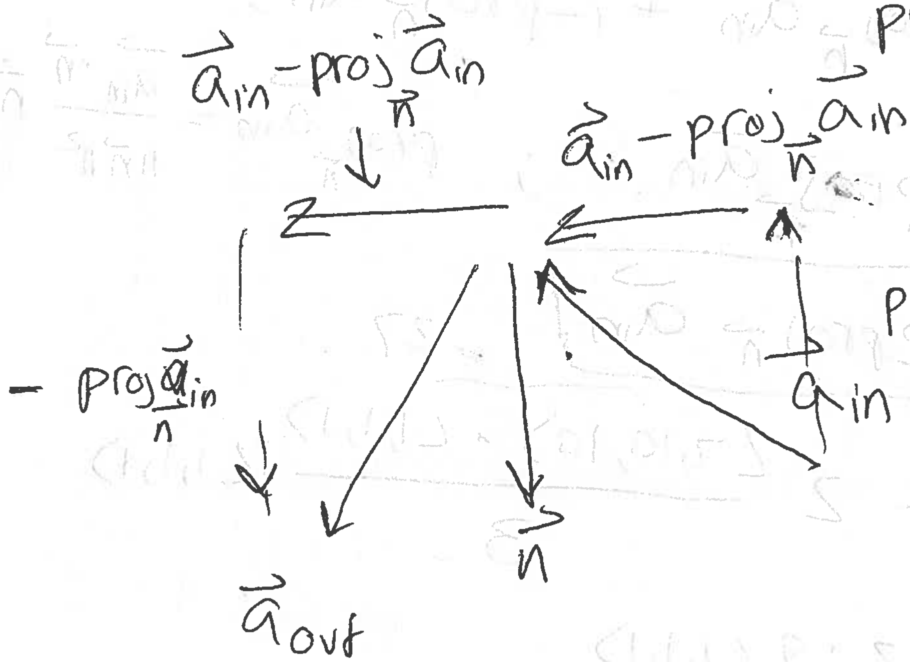
Note that $\vec{n}$ (normal to plane mirror) is:

$\vec{n} = \langle 1,1,1 \rangle$ (computed previously)

also, $\vec{a}_{in} = \overrightarrow{PQ} = \langle 7, 10, 10 \rangle$

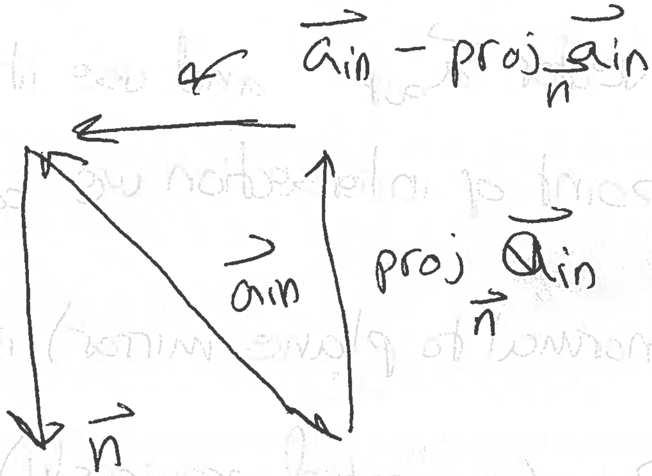


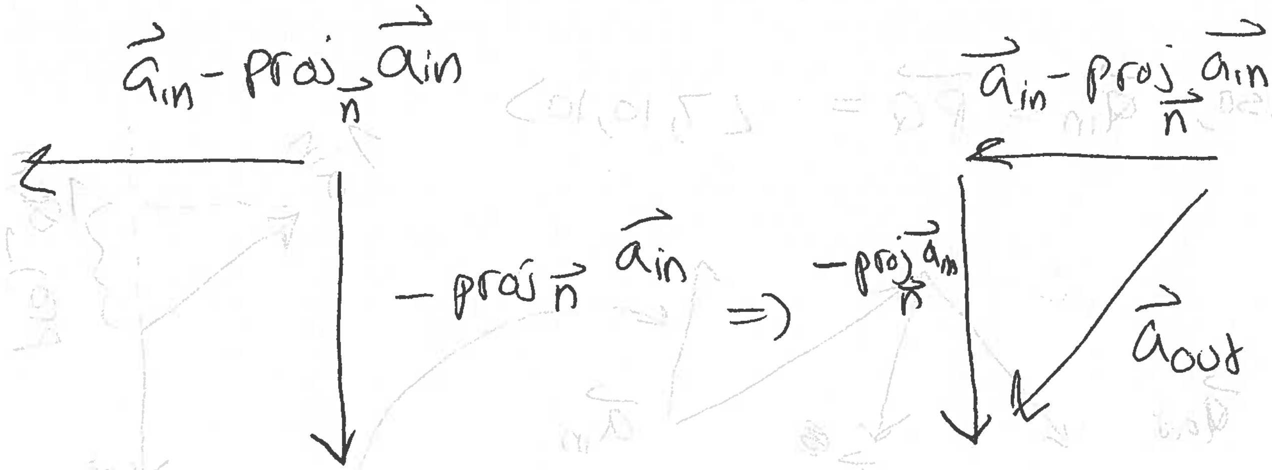
$$proj_{\vec{n}} \vec{a}_{in} = \frac{\vec{a}_{in} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n}$$



must use vector projections because $\left.\begin{array}{l}\vec{a}_{in}, \vec{a}_{out} \\ \vec{n}\end{array}\right\}$ all these vectors are in one plane.

how to calculate $\vec{a}_{out}$?





$$\vec{a}_{out} = \vec{a}_{in} - \text{proj}_{\vec{n}} \vec{a}_{in} + (-\text{proj}_{\vec{n}} \vec{a}_{in})$$

$$= \vec{a}_{in} - 2\,\text{proj}_{\vec{n}} \vec{a}_{in} \; ; \quad \text{proj}_{\vec{n}} \vec{a}_{in} = \frac{\vec{a}_{in} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n}$$

$$\Rightarrow \boxed{\vec{a}_{out} = \vec{a}_{in} - 2\,\text{proj}_{\vec{n}} \vec{a}_{in}}$$

$$= \langle 7,10,10 \rangle - 2 \frac{\overbrace{\langle 7,10,10 \rangle \cdot \langle 1,1,1 \rangle}^{=27}}{3} \langle 1,1,1 \rangle$$

$$= \langle 7,10,10 \rangle - 2 \cdot 9 \langle 1,1,1 \rangle$$

$$= \langle 7,10,10 \rangle - 2\langle 9,9,9 \rangle = \langle -11, -8, -8 \rangle$$

Therefore, given point of intersection

$$\vec{P} = \left\langle \frac{263}{27}, \frac{260}{27}, \frac{260}{27} \right\rangle$$

we have ↙ eqn of reflected light

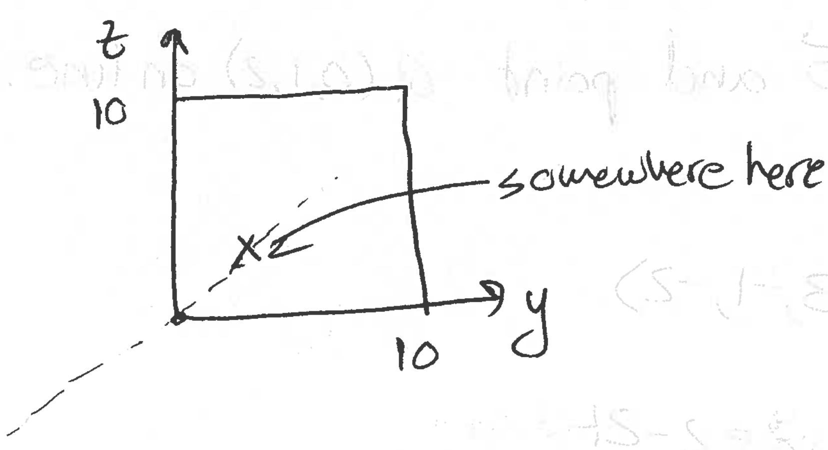
$$\vec{r_2}(t) = \underbrace{\left\langle \frac{263}{27}, \frac{260}{27}, \frac{260}{27} \right\rangle}_{\vec{r_0}} + t\,\underbrace{\langle -11, -8, -8 \rangle}_{\vec{v}}$$

$$= \left\langle \frac{263}{27} - 11t, \frac{260}{27} - 8t, \frac{260}{27} - 8t \right\rangle \quad (*)$$

where does $\vec{r_2}(t)$ intersect?

well if we set $y(t) = 0 \Rightarrow \frac{260}{27} - 8t = 0 \Rightarrow$ solve for t

then $z(t) = y(t) = 0$ and $x(t) < 0$. → goes through back wall! This means $\vec{r_2}(t)$ intersects the back wall: means need to set $x(t) = 0$



get this point by setting $x(t) = 0$!

$$\Rightarrow \frac{263}{27} - 11t = 0$$

$$\Rightarrow t = \frac{263}{297}$$

point of intersection is $\vec{r_2}\left(\frac{263}{297}\right) = \left\langle 0, \frac{28}{11}, \frac{28}{11} \right\rangle$ via (*).

Ex 4 | Find parametric equations for the line through the point $(0,1,2)$ that is parallel to the plane $x+y+z=2$ and perpendicular to the line $x=1+t,\ y=1-t,\ z=2t\ (L)$.

$\Rightarrow$ The line we are finding is $\perp$ to both $\vec{n}=\langle 1,1,1\rangle$ (normal of plane) and the direction vector of L:

$\vec{v}=\langle 1,-1,2\rangle$.

So the direction vector of our line is:

$$\vec{p}=\vec{n}\times\vec{v}=\langle 1,1,1\rangle\times\langle 1,-1,2\rangle=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & -1 & 2\end{vmatrix}=\langle 3,-1,-2$$

We know direction vector $\vec{p}$ and point $Q(0,1,2)$ on line.

Hence:

$$\vec{r}(t)=\langle 0,1,2\rangle+t\langle 3,-1,-2\rangle$$

$$\Rightarrow x=3t\ ;\ y=1-t\ ;\ z=2-2t$$

Ex5 where on the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$

are we after traveling a distance of $\frac{\pi\sqrt{10}}{3}$?

That is, we start at $\vec{A} = \vec{r}(0) = \langle 0, 0, 3 \rangle$

and travel $\frac{\pi\sqrt{10}}{3}$ units along the curve. What are the coordinates of P?

define arclength: $s(t) = \int_0^t \|\vec{r}\,'(u)\|\, du$

$\vec{r}\,'(t) = \langle 2, 6\cos(2t), -6\sin(2t) \rangle$

$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{4 + 36\cos^2(2t) + 36\sin^2(2t)} = \sqrt{4+36} = 2\sqrt{10}$

$\Rightarrow s(t) = \int_0^t 2\sqrt{10}\, du = 2\sqrt{10}\, t$

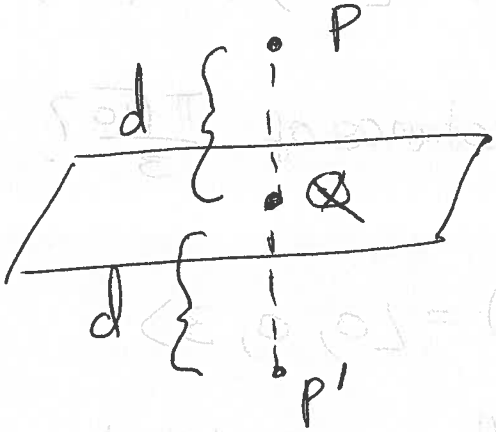
$\Rightarrow$ so time as a function of arc-length is: $t = \frac{s}{2\sqrt{10}}$

$\vec{r}(t(s)) = \langle 2\frac{s}{2\sqrt{10}}, 3\sin\left(\frac{s}{\sqrt{10}}\right), 3\cos\left(\frac{s}{\sqrt{10}}\right) \rangle$

answer is $\vec{r}\left(t\left(\frac{\pi\sqrt{10}}{3}\right)\right) = \langle \frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2} \rangle$

coordinates of P

Ex 7) Reflecting a point in a plane   P reflected into P'.



Q is point on plane closest to P.

$\vec{PQ} \parallel \vec{n}$ of plane

$\vec{P} = \langle 1,2,3\rangle$; plane $\underbrace{2x+3y+5z}_{\vec{n}=\langle 2,3,5\rangle}=3$

eqn of line through P : $\vec{r}(t) = \langle 1,2,3\rangle \mp t\langle 2,3,5\rangle$

Find point Q on plane: $\vec{r}(t) = \langle 1\mp 2t,\ 2\mp 3t,\ 3\mp 5t\rangle$

$\Rightarrow\ 2(1\mp 2t) + 3(2\mp 3t) + 5(3\mp 5t) = 3 \Rightarrow 38t = +20 \Rightarrow t = \frac{10}{19}$

$\vec{r}\left(\frac{10}{19}\right)$ is coordinates of point Q.

$\vec{r}\left(2\cdot\frac{10}{19}\right) = \vec{r}\left(\frac{20}{19}\right)$ are the coordinates of point P'.

also: $P' = P + 2\vec{PQ}$

Ex 8) Find curvature of

$y = f(x)$

$\Rightarrow \vec{r}(x) = x\,\hat{i} + f(x)\,\hat{j}$

$\Rightarrow \vec{r}\,'(x) = \hat{i} + f'(x)\,\hat{j} \Rightarrow \vec{r}\,''(x) = f''(x)\,\hat{j}$

$\vec{r}\,'(x) \times \vec{r}\,''(x) = [\hat{i} + f'(x)\hat{j}] \times [f''(x)\hat{j}]$

$= f''(x)\,\hat{k}$ since $\hat{i}\times\hat{j} = \hat{k}$ and $\hat{j}\times\hat{j} = 0$

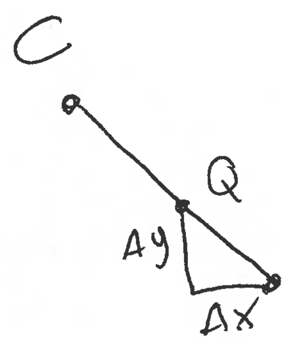
$$K(x) = \frac{\|\vec{r}\,'(x) \times \vec{r}\,''(x)\|}{\|\vec{r}\,'(x)\|^3} = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

$r = \|\vec{PC}\| = 2\sqrt{2}$

Ex 9]

find C given P and r and slope of line $= -1$.

traveling from P to C knowing direction and distance



$P(1, \frac{1}{2})$

can find that $Q(0, \frac{3}{2})$ is on the line.

unit direction vec

$\vec{PQ} = \langle 0-1, \frac{3}{2} - \frac{1}{2} \rangle = \langle -1, 1 \rangle \Rightarrow \vec{v} = \frac{\vec{PQ}}{\|\vec{PQ}\|} = \frac{\langle -1, 1 \rangle}{\sqrt{2}} = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

$\Rightarrow C = (1, \frac{1}{2}) + 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \langle -1, \frac{5}{2} \rangle$

Ex) when is $\vec{r}(t) = \langle t^2, t^2, t^2 \rangle$ closest to $\langle -1,-1,-1 \rangle$? (i.e. at which point?)

$(-1,-1,-1)$ is never on $\vec{r}(t)$ since $t^2 \geq 0$.

$\Rightarrow \vec{r} - \vec{p} = \langle t^2+1, t^2+1, t^2+1 \rangle$

$\|\vec{r}-\vec{p}\|^2 = 3(t^2+1)^2$

$\Rightarrow \dfrac{d}{dt}\left[\|\vec{r}-\vec{p}\|^2\right] = 6(t^2+1)\cdot 2t = 0 \Rightarrow t = 0$

closest point is $\vec{r}(0) = \langle 0,0,0 \rangle$

(can use second deriv to check dist is min)