Review #1

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limits

$\lim_{(x,y)\to(0,0)} f(x,y)$ exists or not?

Ex) $f(x,y) = \begin{cases} \dfrac{xy}{x^2-y^2} & \text{if } x \neq \pm y \\ 0 & \text{if } x = \pm y \end{cases}$

$\Rightarrow$ approach along $y = \alpha x$

$\Rightarrow f(x, \alpha x) = \dfrac{\alpha x^2}{x^2 - \alpha^2 x^2} = \dfrac{\alpha}{1-\alpha^2}$

$f(x,2x) = -\frac{2}{3}$ and $f(x,-2x) = +\frac{2}{3}$

$\Rightarrow f(x,y)$ has different constant values on straight lines through zero so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Ex) $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$ Let's do save trick

$y = \alpha x \Rightarrow \dfrac{3x^2\alpha x}{x^2+\alpha^2x^2} = \dfrac{3\alpha x^3}{x^2(1+\alpha^2)} = \dfrac{3\alpha x}{1+\alpha^2}$

as $x \to 0$ this goes to zero for any α
so we suspect this limit may exist.

Recall useful property:

If $|f(x,y) - L| \le g(x,y)$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = 0$

then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$. (see Corral book)

we suspect $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0$. Let's use

above property to check:

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| \le g(x,y)$$ & we need to find $g(x,y)$

Note that $\dfrac{x^2}{x^2+y^2} \le \dfrac{x^2+y^2}{x^2+y^2} = 1$

$$\Rightarrow \left| \frac{3x^2y}{x^2+y^2} \right| = \frac{|3|\cdot|x^2|\cdot|y|}{|x^2+y^2|} \le 3|y| = 3y^2 \le 3\sqrt{x^2+y^2}$$

so $g(x,y) = 3\sqrt{x^2+y^2}$

But $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0$ so this proves our limit is 0.

Continuity

$$f(x,y) = \frac{\sin(\pi(x^2+y^2))}{x^2+y^2}$$

$\Rightarrow r^2 = x^2+y^2$ (polar coordinates)

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\sin(\pi(x^2+y^2))}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(\pi r^2)}{r^2} = \lim_{r\to 0} \frac{2\pi r\cos(\pi r^2)}{2r}$$

$$= \pi$$

However, f not continuous at $(0,0)$. For continuity we need

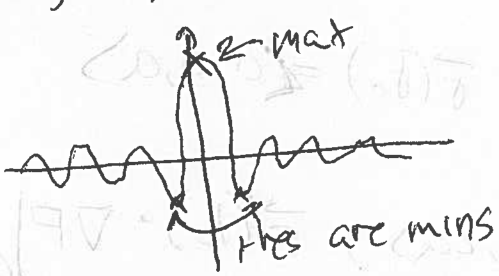
$$f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$$

our limit exists, but $f(0,0)$ does not exist!

domain and range of $f(x,y)$:

domain is $\mathbb{R}^2 \setminus \{0,0\}$ (all (x,y) other than $x=0, y=0$)

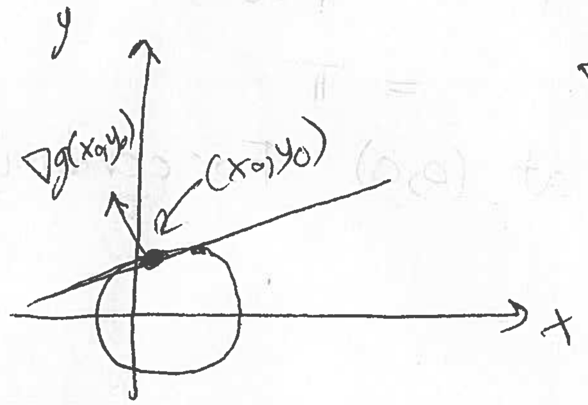
for range, it is simpler to plot function $f(r) = \frac{\sin(\pi r^2)}{r^2}$ and to report min, max of $f(r)$.



Gradient

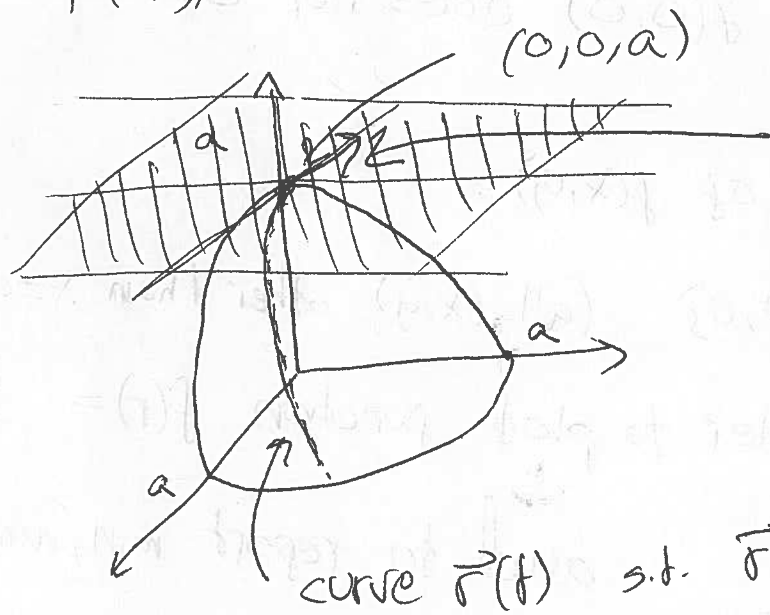
The gradient is perpendicular to level curves. Let's look at what this means in 2D and 3D.

2D: $g(x,y) = \underbrace{x^2+y^2-4x = 1}_{\text{this defines a circle ("level curve")}}$



$\nabla g(x_0,y_0) \perp$ to tangent line at (x_0,y_0).

3D: $F(x,y,z) = x^2+y^2+z^2 = a^2$ (sphere of radius a)



$\vec{r}\,'(t_0)$

$\nabla F \perp$ to curve passing through some point (∇F tangent $\perp$ to its tangent vector at that same point)

curve $\vec{r}(t)$ s.t. $\vec{r}(t_0) = \langle 0,0,a \rangle$

$\nabla F = \langle 2x, 2y, 2z \rangle \Rightarrow \nabla F|_{(0,0,a)} = \langle 0,0,2a \rangle$; $\vec{r}\,'(t_0) \cdot \nabla F|_{(0,0,a)} = 0$.

Exam 2 from Spring 2014 (on exam archive website)

#2) Bonnie flies along path $\vec{r}(t)$. Temperature distribution is $T(x,y)$ and $T(1,3)=10$. Bonnie flies at speed $\sqrt{5}$.

(a) As Bonnie flies North, over $(1,3)$ she notices that $\frac{dT}{dt} = 2\sqrt{5}$. However East over $(1,3)$, $\frac{dT}{dt} = 5\sqrt{5}$.

Calculate $\nabla T|_{(1,3)}$.

North : $\hat{j}$; East : $\hat{i}$

$\vec{u} = \hat{j}$; $\vec{v} = \hat{i}$

$D_{\vec{u}}T = \nabla T \cdot \vec{u}$

$D_{\vec{v}}T = \nabla T \cdot \vec{v}$

directional derivatives

$\Downarrow$

$T(x,y) \Rightarrow \nabla T = \langle T_x, T_y \rangle$

$\nabla T \cdot \vec{u} = \langle T_x, T_y \rangle \cdot \langle 0,1 \rangle$

Notice however that we want $\frac{dT}{dt}$. $(x=x(t), y=y(t))$

$\Downarrow$

$$\Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = \langle T_x \cdot T_y \rangle \cdot \underbrace{\langle x'(t), y'(t) \rangle}_{v(t)} = \underbrace{\nabla F \cdot (\|\vec{v}\| \vec{u})}_{\text{this is a scaled dir derivative}}$$

$\vec{v}(t) = \|\vec{v}(t)\|(\text{direction unit vector}) = \|\vec{v}(t)\|\vec{u}$

in North direction $(\vec{u} = \hat{j}) \Rightarrow \frac{dT}{dt} = \langle T_x, T_y \rangle \cdot \langle 0,1 \rangle \sqrt{5} = 2\sqrt{5}$

$\vec{T}'(t) = \nabla T \cdot (\underbrace{\|\vec{v}\|}_{\text{speed}} \underbrace{\vec{u}}_{\text{direction unit vector}})$

$\Rightarrow \sqrt{5}\, T_y = 2\sqrt{5} \Rightarrow T_y = 2$

East: $\frac{dT}{dt} = \langle T_x, T_y \rangle \cdot \left(\underbrace{\sqrt{5}}_{\text{speed}} \underbrace{\langle 1, 0 \rangle}_{\text{unit direction vecto}} \right) = 5\sqrt{5}$

$\Rightarrow \sqrt{5}\, T_x = 5\sqrt{5} \Rightarrow T_x = 5$

$\Rightarrow \nabla T = \langle T_x, T_y \rangle = \langle 5, 2 \rangle$ at point $(1,3)$.

(b) If Bonnie continues on her Eastern path $\vec{r}(t)$ for short time interval $\overset{=0.1}{\Delta t}$, by approximately how much does the temperature change.

$\Rightarrow$ We want to approximate ΔT using dT.

$T = T(x,y) = T(x(t), y(t))$ = function of time t!

$\Rightarrow dT = \frac{dT}{dt} dt \Rightarrow$ take $dt = \Delta t \Rightarrow dT = \frac{dT}{dt} \Delta t$

$\frac{dT}{dt}$ in East at $(1,3)$ is $5\sqrt{5}$ and $\Delta t = 0.1$

$\Rightarrow dT = (5\sqrt{5})(0.1) = \frac{\sqrt{5}}{2}$

(c) As Bonnie flies past (1,3) heading East, at what rate was T changing wrt distance traveled, $\frac{dT}{ds}$?

$$\Rightarrow \frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds}$$

but recall that $\frac{ds}{dt} = \|\vec{r}\,'(t)\|$
$= \text{speed}$

$$\Rightarrow \frac{ds}{dt} = \sqrt{5} \Rightarrow \frac{dt}{ds} = \frac{1}{\sqrt{5}}$$

; $\frac{dT}{dt}$ at (1,3) in East direction is $5\sqrt{5}$

$$\Rightarrow \frac{dT}{ds} = 5\sqrt{5}\left(\frac{1}{\sqrt{5}}\right) = 5$$

$\left[\text{i.e. } \frac{dT}{ds} = \frac{dT/dt}{ds/dt}\right]$

(d) Bonnie wants to fly from (1,3) towards (0,0). As she leaves (1,3) at what rate is T changing wrt distance traveled, $\frac{dT}{ds}$?

$$\boxed{\frac{dT}{ds} = \frac{dT/dt}{ds/dt}}$$

$$\frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds}$$

we know $\frac{ds}{dt}$ but we do not know $\frac{dT}{dt}$ at (1,3) in given direction $\Rightarrow$ must calculate this.

direction: $\vec{v} = \langle 0-1, 0-3\rangle = \langle -1,-3\rangle \Rightarrow$ unit: $\frac{\vec{v}}{\|\vec{v}\|} = \frac{-1}{\sqrt{10}}\langle 1,3\rangle$

$$\Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = \nabla T\Big|_{(1,3)} \cdot \underbrace{\vec{v}(t)}_{\|\vec{v}(t)\|\vec{u}} = \langle 5,2\rangle \cdot \sqrt{5}\left(\frac{-1}{\sqrt{10}}\right)\langle 1,3\rangle$$

$$= -\frac{1}{\sqrt{2}}(5+6) = \frac{-11}{\sqrt{2}}$$

Thus,

$$\frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds} = \frac{-11}{\sqrt{2}} \cdot \frac{1}{\sqrt{5}} = \frac{-11}{\sqrt{10}}$$

#3) The function f depends on the variables x and y. In addition, x depends on r and s. However, y only depends on r. For a specific set of values of r and s it is known that $\frac{\partial f}{\partial x} = 5$, $\frac{\partial f}{\partial y} = 6$ while $\frac{\partial x}{\partial r} = 2$, $\frac{\partial x}{\partial s} = 3$, and $\frac{dy}{dr} = 3$.

(a) If x is increased by 0.1 and y is constant, how much does f change?

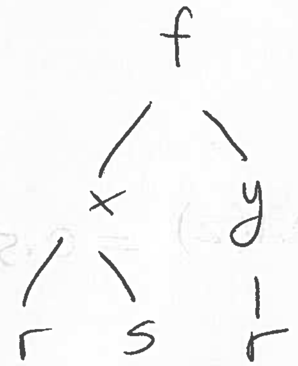
$f = f(x,y) \Rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$; use $dx = \Delta x$, $dy = \Delta y$

$$\Delta f \approx df = \frac{\partial f}{\partial x}\Delta x + 0 = 5(0.1) = 0.5$$

(b) If both x and y decrease by 0.1, how much does f change?

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = 5(-0.1) + 6(-0.1) = -1.1$$

(c) If s is held constant and r increases by 0.1, how much does f change?



$\Delta s = 0$; $\Delta r = 0.1$

$$\Delta f \approx \partial f = f_x\, dx + f_y\, dy$$

take $dx = \Delta x$, $dy = \Delta y$

But $\Delta x \approx dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial s} ds$

$$= \frac{\partial x}{\partial r} \Delta r + \frac{\partial x}{\partial s} \Delta s$$

$$\Delta f \approx f_x \Delta x + f_y \Delta y$$

But $\Delta x \approx x_r \Delta r + x_s \underbrace{\Delta s}_{=0} = 2(0.1) = 0.2$

$\Delta y \approx y_r \Delta r + y_s \underbrace{\Delta s}_{=0} = 3(0.1) = 0.3$

$\Rightarrow \Delta f \approx 5(0.2) + 6(0.3) = 2.8$

(d) If r is held constant and s decreases by 0.1, how much does f change?

$\Delta r = 0$, $\Delta s = -0.1$

$\Rightarrow \Delta x \approx x_s \Delta s = 3(-0.1) = -0.3$

$\Delta y \approx y_s \Delta s = 0$ ($\frac{\partial y}{\partial s} = 0$ since y depends only on r)

$\Rightarrow \Delta f \approx$ ~~[crossed out]~~ $f_x \Delta x + f_y \Delta y = f_x(-0.3) = 5(-0.3) = -1.5$

(e) If s and r both increase by 0.1, how much does f change?

$\Rightarrow \Delta s = 0.1$ and $\Delta r = 0.1$

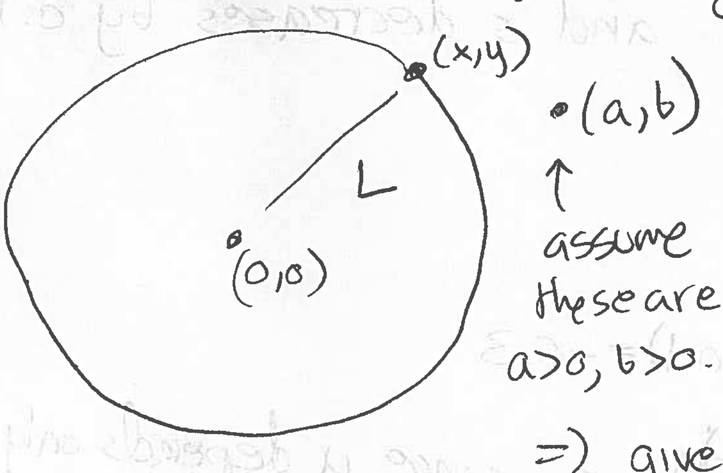
$\Rightarrow \Delta x \approx x_r \Delta r + x_s \Delta s = 0.1(3) + 0.1(2) = 0.5$

$\Delta y \approx y_r \Delta r + \underbrace{y_s \Delta s}_{=0} = 0.1(3) = 0.3$

$\Rightarrow \Delta f \approx f_x \Delta x + f_y \Delta y = 5(0.5) + 6(0.3) = 0.25 + 1.8$

$= 2.05$

#4) A dog is tied to a rope of length L at the origin. A squirrel is teasing the dog by sitting at (a,b). The dog is trying to get as close as possible to the squirrel. Determine the location of the dog.



dog is at (x,y)
(x,y) is on circle
distance between (x,y) and (a,b) to be minimized

$\Rightarrow$ given above set up Lagrange multiplier system.

6

$$f(x,y) = d^2 = (x-a)^2 + (y-b)^2$$ to minimize (function)

$$g(x,y) = x^2 + y^2 = L^2$$ (constraint)

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-a), 2(y-b) \rangle = \lambda \langle 2x, 2y \rangle$$

$$x - a = \lambda x \quad \text{and} \quad y - b = \lambda y$$

$\Rightarrow$ solve for $\lambda \Rightarrow \lambda = \dfrac{x-a}{x} = \dfrac{y-b}{y}$

$$\Rightarrow \frac{(x-a)y}{xy} = \frac{(y-b)x}{xy} \Rightarrow (x-a)y = (y-b)x$$

$$xy - ay = xy - bx$$

$\Rightarrow ay = bx \Rightarrow y = \dfrac{b}{a}x$ (this is relation between x and y at constrained point)

Plug into constraint:

$$g(x,y) = x^2 + y^2 = L = x^2 + \left(\frac{b}{a}x\right)^2 \Rightarrow x^2 + \frac{b^2}{a^2}x^2 = L^2$$

$$\Rightarrow x^2\left(1 + \frac{b^2}{a^2}\right) = L^2 \Rightarrow x^2 = \frac{L^2}{\left(1 + \frac{b^2}{a^2}\right)} = \frac{a^2L^2}{a^2+b^2}$$

assume $a > 0$, $b > 0$ (so we can choose + root of x):

$$\Rightarrow x = \frac{aL}{\sqrt{a^2+b^2}} \quad \text{and} \quad y = \frac{b}{a}x = \frac{bL}{\sqrt{a^2+b^2}}$$

(x,y) is the location of the dog when squirrel is at (a,b)

#5] Consider the function $f(x,y) = \frac{x^2}{2} + \frac{x^3}{3}(1-y)$

(a) Determine the linearization of $f(x,y)$ about $(1,1)$.

$\Rightarrow$ linearization is just the linear part of Taylor polynomial.

$$f(x,y) \approx \underbrace{f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}_{L(a,b)}$$

$$L(1,1) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$f(1,1) = \frac{1^2}{2} + \frac{1^3}{3}(1-1) = \frac{1}{2}$$

$$f_x = \frac{2x}{2} + (1-y)\frac{3x^2}{3} = x + (1-y)x^2$$

$f_x(1,1) = 1$;

$$f_y = -\frac{x^3}{3} \Rightarrow f_y(1,1) = -\frac{1}{3}$$

$$\Rightarrow L(1,1) = \frac{1}{2} + (x-1) - \frac{1}{3}(y-1)$$

(b) approximate $f(1.1, 1.2)$ using linearization in (a):

$$f(1.1, 1.2) \approx L(1.1, 1.2) = \frac{1}{2} + (1.1-1) - \frac{1}{3}(1.2-1)$$

$$= \frac{1}{2} + (0.1) - \frac{1}{3}(0.2)$$

#1) consider $f(x,y) = 10 + x^2 + \frac{y^2}{2} + x^2(1-y)$

(a) Find all critical pts of $f(x,y)$ and classify them

$\Rightarrow \nabla f = \langle f_x, f_y \rangle = \langle 2x(2-y), y - x^2 \rangle = \langle 0, 0 \rangle$

$\Rightarrow \underbrace{x(2-y) = 0}$ and $y = x^2$

either $x = 0$ or $y = 2$

case 1 : $x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

case 2 : $y = 2 \Rightarrow 2 = x^2 \Rightarrow x = \pm\sqrt{2} \Rightarrow (\sqrt{2}, 2), (-\sqrt{2}, 2)$

$f_{xx} = 4$; $f_{xy} = 2x$; $f_{yy} = 1$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2 = 4 - (2x)^2 = 4 - 4x^2 = 4(1 - x^2)$$

$D(0,0) = 4$; $D(\pm\sqrt{2}, 2) = 4(1-2) = -4 < 0$

So $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$ are saddle points.

$(0,0)$ is either local min or local max, let's check :

$f_{xx}(0,0) = 4 > 0 \Rightarrow (0,0)$ is a local min.

(b) Determine value at each critical point

$$\Rightarrow f(0,0) = 10$$

$$f(\sqrt{2},2) = f(-\sqrt{2},2) = 10 + 2 + 2 + 2(1-2) = 12$$

c) Determine std linearization of $f(x,y)$ of the functions extreme values.

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$= f(a,b) + \underbrace{\langle f_x, f_y \rangle \Big|_{(a,b)}}_{=0 \text{ at critical points}} \cdot \langle x-a, y-b \rangle$$

$$L(x,y) = f(0,0) = 10$$

note: only $(a,b) = (0,0)$ is extreme ~~value~~ point (local min), rest are saddle points.

Summer 2014 Exam 2

(on APPM 2350 exam archive website)

#4) Take $f(x,y) = x^2 + xy^3$

(a) Find first order Taylor expansion about $(2,1)$

$\Rightarrow$ This is just first term of Taylor series (the linear approximation)

$$\Rightarrow f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = L(x,y)$$

with $(a,b) = (2,1)$.

$f(2,1) = 6$; $f_x = 2x + y^3 \Rightarrow f_x(2,1) = 5$

$f_y = 3xy^2 \Rightarrow f_y(2,1) = 6$

$$\Rightarrow L(x,y) = 6 + 5(x-2) + 6(y-1)$$

b) Approximate $f(1.9, 1.1)$

$1.9 \le x \le 2.1$
$0.9 \le y \le 1.1$

$f(1.9, 1.1) \approx L(1.9, 1.1) = \frac{61}{10}$ (just plug in)

c) Bound the error on the approximation if $|x-2| \le 0.1$ and $|y-1| \le 0.1$.

Again, the error by Taylor's theorem is just the second term of Taylor series evaluated at some point between (a,b) and (x,y).

To bound the error, we just need to bound all the terms in the second term of Taylor series.

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$+ \underbrace{\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]}_{\text{we must put bounds on all these terms}}$$

we have $|x-2| \le 0.1$ and $|y-1| \le 0.1$ just need bounds on $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$. partials evaluated at some (c,d) → any pt between (a,b) and (x,y) → need to bound in this interval.

$f_{xx} = 2 \Rightarrow |f_{xx}| \le 2$

$f_{xy} = 3y^2 \Rightarrow$ max occurs when $y = 1+0.1 = 1.1$

(remember we are considering interval from (a,b) to (x,y), but if you can bound these for all (x,y) such as for f_{xx} it works too).

(c,d) between (a,b) and (x,y)

$\Rightarrow |f_{xy}(c,d)| \le 3(1.1)^2 \sim 3.63$

$f_{yy} = 6xy \Rightarrow$ max when $x = 2+0.1$ and $y = 1+0.1$

$\Rightarrow |f_{yy}(c,d)| \le 6(2.1)(1.1) \sim 13.86$

Hence an upper bound on error (of using just first term of Taylor series instead of all terms is):

$$\text{error} \leq \frac{1}{2}\left[(0.1)^2 \cdot 2 + 2\cdot(0.1)^2 \cdot 3.63 + (0.1)^2 \cdot 13.86 \right]$$

$$\sim 0.12$$

notice: one can also obtain worser error bounds higher and less tight). For example, you can use:

$$|f_{xx}|, |f_{yy}|, |f_{xy}| \leq \overset{=M}{13.86}, \text{ etc.}$$

then

$$E \leq \frac{1}{2} M \left(|x-a|^2 + |y-b|^2 \right)$$ ← this works too but is a less tight bound then above.

Note: If you want upper bound on quadratic approximation (first two terms of Taylor series), then you must bound all terms in 3rd term of Taylor series i.e:

$$\frac{1}{3!}\left[(x-a)^3 f_{xxx}(c,d) + 3(x-a)^2(y-b) f_{xxy}(c,d) + 3(x-a)(y-b)^2 f_{xyy}(c,d) + (y-b)^3 f_{yyy}(c,d) \right]$$

#3 Gary the goat is wondering around a field eating hay. The hay density is given by

$H(x,y) = \frac{1}{\sqrt{1+x^2+y^2}}$. Gary walks around the path given by $\vec{r}(t) = \langle t\cos t, t\sin t\rangle$ for $t \geq 0$.

(a) what is $\frac{dH}{dt}$ at $t=2$?

$H = H(x(t), y(t))$

$$\Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} = \nabla H \cdot \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$$

$$= \nabla H \cdot \vec{r}\,'(t)$$

$\vec{r}\,'(t) = \langle \cos t - t\sin t, \sin t + t\cos t \rangle$

$$\nabla H = \langle \frac{-x}{(1+x^2+y^2)^{3/2}}, \frac{-y}{(1+x^2+y^2)^{3/2}} \rangle$$

$$\nabla H(x(t), y(t)) = \langle \frac{-t\cos t}{(1+t^2)^{3/2}}, \frac{-t\sin t}{(1+t^2)^{3/2}} \rangle$$

$$\Rightarrow \frac{dH}{dt} = \nabla H(x(t), y(t)) \cdot \vec{r}\,'(t) = -\frac{1}{(1+t^2)^{3/2}}\left(t\cos^2 t - \cancel{t^2 \sin t\cos t} + t\sin^2 t + \cancel{t^2 \sin t\cos t}\right)$$

$$= \frac{-t}{(1+t^2)^{3/2}}$$

So $\left.\frac{dH}{dt}\right|_{t=2} = \frac{-2}{5^{3/2}}$

(b) A $t=2$, what is the slope (directional derivative) of the hay density in the direction that Gary is moving?

Notice that for functions of two variables, the directional derivative for some direction in xy-plane can be interpreted as a slope.

$$D_{\vec{T}} H = \nabla H \cdot \vec{T} = \nabla H \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$\vec{T}$ ← unit tangent vector

Tangent vector gives us the unit vector where Gary is moving, in which direction at each time t. This is the direction vector for our directional derivative.

$$\|\vec{r}\,'(t)\| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \dots = \sqrt{1+t^2}$$

$$D_{\vec{T}} H = \left\langle \frac{-t\cos t}{(1+t^2)^{3/2}}, \frac{-t\sin t}{(1+t^2)^{3/2}} \right\rangle \cdot \frac{1}{\sqrt{1+t^2}} \langle \cos t - t\sin t, \sin t + t\cos t \rangle$$

$$= \dots = \frac{-t}{(1+t^2)^2} \rightarrow \left. D_{\vec{T}} H \right|_{t=2} = \frac{-2}{5^2} = -\frac{2}{25}$$

(c) At $t=5$, what direction should Gary walk in if he wants to walk in the direction of greatest increase of hay density.

The direction of greatest increase of $H(x,y)$ is in the direction of the gradient. The direction of greatest decrease of $H(x,y)$ is in direction of minus (negative of) the gradien of H.

for greatest increase, walk towards $\nabla H(x(5), y(5))$

$$\nabla H(x(5), y(5)) = \left\langle -\frac{5\cos 5}{26^{3/2}}, -\frac{5\sin 5}{26^{3/2}} \right\rangle$$

for greatest decrease, walk towards $-\nabla H(x(5), y(5))$.

#1] $F(u,v) = F(u(x), v(x,y))$

$$\Rightarrow \frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\underbrace{\frac{\partial u}{\partial y}}_{=0} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} \quad \text{etc}$$

Exam 2, Fall 2013

#1] $f(x,y) = 10 + \dfrac{y^2}{2} + x^2(1-y)$

surface on which marble rolls

(c) where can marble come to rest?

$\Rightarrow$ at points which are local min

$\Rightarrow \nabla f(x,y) = 0$ and $D > 0$ and $f_{xx} > 0$.

$\Rightarrow \nabla f = \langle 2x(1-y),\ y - x^2 \rangle = \langle 0, 0 \rangle$

critical pts

$2x(1-y) = 0 \Rightarrow \left.\begin{array}{l} x = 0 \Rightarrow y = x^2 = 0 \\ y = 1 \Rightarrow 1 = x^2 \Rightarrow x = \pm 1 \end{array}\right\}$ $(0,0)$, $(1,1)$, $(-1,1)$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

$f_{xx} = 2(1-y)$; $f_{yy} = 1$; $f_{xy} = -2x$

$\rightarrow D(x,y) = 2(1-y)\cdot 1 - (-2x)^2 = 2(1-y) - 4x^2$

$\left.\begin{array}{l} D(0,0) = 2 - 0 = 2 > 0 \\ f_{xx}(0,0) = 2(1-0) = 2 > 0 \end{array}\right\}$ (0,0) is a local min

must check all other pts too

#2) You are trying to estimate the volume V of a circular cylinder by measuring the radius r and height h. The relative error $\frac{\Delta V}{V}$ must be less than 0.01. Estimate $\frac{\Delta r}{r}$ and $\frac{\Delta h}{h}$ if the relative error in the height is equal to twice the relative error in the radius.

the last statement says: $\frac{\Delta h}{h} = 2\,\frac{\Delta r}{r}$

$$\Delta V \approx dV = \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h \qquad \left(\text{taking } \begin{matrix} dr = \Delta r \\ dh = \Delta h \end{matrix}\right)$$

$V(r,h) = \pi r^2 h$ (volume of cylinder)

$$\Rightarrow \Delta V \approx 2\pi r h\,\Delta r + \pi r^2 \Delta h$$

$$\frac{\Delta V}{V} = \frac{2\pi r h\,\Delta r}{\pi r^2 h} + \frac{\pi r^2 \Delta h}{\pi r^2 h} = 2\,\frac{\Delta r}{r} + \frac{\Delta h}{h}$$

$$= 2\left(\frac{1}{2}\,\frac{\Delta h}{h}\right) + \frac{\Delta h}{h} = \frac{2\Delta h}{h}$$

note: for relative or percent errors use $\frac{\Delta V}{V}$, not ΔV

$$\frac{\Delta V}{V} \le 0.01 \Rightarrow \frac{2\Delta h}{h} \le 0.01 \Rightarrow \frac{\Delta h}{h} \le 0.005$$

and $\frac{\Delta r}{r} \le \frac{1}{2}(0.005) = 0.0025$

Ex)

Find an equation of the tangent plane to

$z = f(x,y) = e^{-(x^2+y^2)}$ at $(\frac{1}{2}, \frac{1}{3})$.

$f(x,y)$ is a surface but it is given as a level curve of the function $F(x,y,z) = e^{-(x^2+y^2)} - z = 0$ and ∇F is $\perp$ to level curves.

∇F is $\perp$ to ~~some~~ any curve $\vec{r}(t)$ that goes through some point (a,b,c) on surface of level curve $F(x,y,z) = 0$ (i.e. $\nabla F(a,b,c) \perp \vec{r}\,'(t_0)$ where $\vec{r}(t_0) = \langle a,b,c \rangle$).

The point $(a,b,c) = \left(\frac{1}{2}, \frac{1}{3}, e^{-(\frac{1}{4}+\frac{1}{9})}\right)$

$$\Rightarrow \nabla F(a,b,c) = \left\langle e^{-(x^2+y^2)}(-2x),\ e^{-(x^2+y^2)}(-2y),\ -1 \right\rangle\Big|_{(a,b,c)}$$

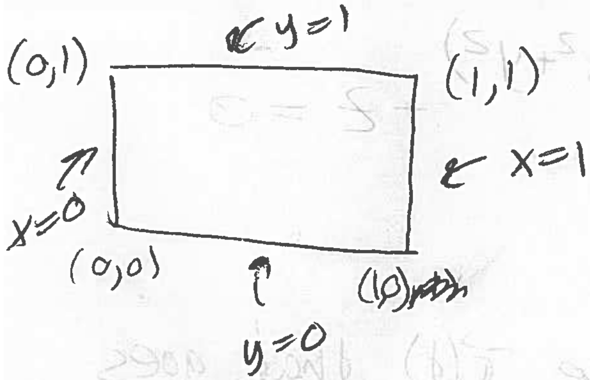
$$= \left\langle e^{-(\frac{1}{4}+\frac{1}{9})},\ \frac{2}{3}e^{-(\frac{1}{4}+\frac{1}{9})},\ -1 \right\rangle$$

eqn of tangent plane: $\nabla F(a,b,c) \cdot \langle x-a, y-b, z-c \rangle = 0$

$$\Rightarrow e^{-(\frac{1}{4}+\frac{1}{9})}\left(x-\frac{1}{2}\right) + \frac{2}{3}e^{-(\frac{1}{4}+\frac{1}{9})}\left(y-\frac{1}{3}\right) - \left(z - e^{-(\frac{1}{4}+\frac{1}{9})}\right) = 0$$

Ex) $f(x,y) = x^3 + y^2 - 3x - 2y$

Find absolute min/max in the rectangle with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$



we must check vals f on bdry and at critical pts inside rectangle.

$f_x = 3x^2 - 3$; $f_y = 2y - 2 \Rightarrow y = 1$

$f_x = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow$ $x = 1, y = 1$; $x = -1, y = 1$

$\Rightarrow \underbrace{(-1,1)}_{\text{not inside rectangle}}$ and $(1,1)$

on boundaries: $f(x,0) = x^3 - 3x = g_1(x) \Rightarrow g_1'(x) = 3x^2 - 3 = 0$

$f(x,1) = x^3 - 3x - 1 = g_2(x) \Rightarrow g_2'(x) = 3x^2 - 3 = 0$

max is 0 at $(0,0)$ and min is -3 at $(1,1)$.

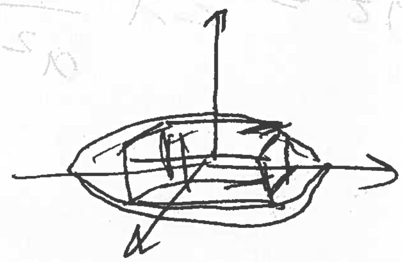
$f(0,y) = y^2 - 2y = g_3(y) \Rightarrow g_3'(y) = 2y - 2 = 0$

$f(1,y) = -2 + y^2 - 2y \Rightarrow g_4(y) \Rightarrow g_4'(y) = 2y - 2 = 0$

we must check f at $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$!

Ex) Find the box of largest volume that is inside ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Let (x,y,z) all > 0 be corner of box.

It has to be on ellipsoid along with all other corners. So corners must satisfy eqn of ellipsoid.

dimensions: $2x,\ 2y,\ 2z$

$V(x,y,z) = (2x)(2y)(2z) = 8xyz$ (volume function)

$g(x,y,z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ (constraint)

$$\nabla V = \lambda \nabla g$$

$$\Rightarrow \left.\begin{aligned} 8yz &= 2\lambda \frac{x}{a^2} \\ 8zx &= 2\lambda \frac{y}{b^2} \\ 8xy &= 2\lambda \frac{z}{c^2} \end{aligned}\right\}$$

$$8xyz = 2\lambda \frac{x^2}{a^2}$$

$$8xyz = 2\lambda \frac{y^2}{b^2}$$

$$8xyz = 2\lambda \frac{z^2}{c^2}$$

$$\Rightarrow 2\lambda \underbrace{\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right]}_{=1 \text{ (recall constraint!)}} = 24xyz \Rightarrow \lambda = 12xyz$$

Then, ← plugin for λ

$$8xyz = 2\lambda \frac{x^2}{a^2} = 2 \cdot 12xyz \cdot \frac{x^2}{a^2}$$

$$= \frac{24x^3yz}{a^2}$$ then mult both sides by a^2

$$\Rightarrow \quad yz(a^2 - 3x^2) = 0$$

Similarly,

$$zx(b^2 - 3y^2) = 0 \quad \text{and} \quad xy(c^2 - 3z^2) = 0$$

Since we assumed, $x > 0, y > 0, z > 0$:

$$x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}$$

$$V_{max} = \frac{8abc}{(\sqrt{3})^3}$$

Review (continued)

10/21/2014

Ex) (Spring 2013, #3)

Bonnie flies path $\vec{r}(t)$ in a field. In this field temperature distribution is $T(x,y)$. At some time t^*, we have $\vec{r}(t^*) = 1\hat{i} + 3\hat{j}$; $\vec{v}(t^*) = 2\hat{i} + 1\hat{j}$; and $\vec{a}(t^*) = 3\hat{i} + 2\hat{j}$. Furthermore, at $\vec{r}(t^*) = \langle 1, 3 \rangle$:

$\nabla T(1,3) = 2\hat{i} + 5\hat{j}$ and $T(1,3) = 10$.

(a) As Bonnie flies past $\vec{r}(t^*)$, what is $\frac{dT}{dt}$?

$T = T(x,y) = T(x(t), y(t))$ $\Rightarrow$ ultimately a function of time.

$$\frac{dT}{dt} = \left(\frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} \right) = \langle T_x, T_y \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$
$$= \nabla T \cdot \vec{r}\,'(t)$$

$$\left.\frac{dT}{dt}\right|_{t^*} = \nabla T(x(t^*), y(t^*)) \cdot \vec{r}\,'(t^*) = \langle 2,5 \rangle \cdot \langle 2,1 \rangle = 4+5 = 9$$

(b) As she flies past $\vec{r}(t^*)$, what is $\frac{dT}{ds}$?

$$\frac{dT}{ds} = \frac{dT/dt}{ds/dt}$$

$$s(t) = \int_0^t \|\vec{r}\,'(t)\|\,dt \Rightarrow \frac{ds}{dt} = \|\vec{r}\,'(t)\| = \|\vec{v}(t)\|$$

$$\left.\frac{dT}{ds}\right|_{t^*} = \frac{1}{\|\vec{v}(t^*)\|} \left.\frac{dT}{dt}\right|_{t^*} = \frac{1}{\sqrt{5}} \cdot 9 = \frac{9}{\sqrt{5}}$$

(c) If Bonnie continues on her path $\vec{r}(t)$ from t^*, ~~to~~ for a short interval $\Delta t = 0.1$, by how approx. much does the temperature change?

$$T = T(x(t), y(t))$$

$$\Delta T \approx dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy = \frac{dT}{dt}dt = \frac{dT}{dt}\Delta t = 9 \cdot 0.1 = 0.9$$

at $t = t^*$.

(d) If at time t^*, Bonnie sees her favorite flower and flies towards it in the direction of the greatest rate of increase of T. Assuming she maintains her same speed, by how much does T change after she flies for $\Delta t = 0.1$

Direction of greatest increase of T = parallel to $\nabla T(x(t^*), y(t^*))$

$\nabla T|_{t^*} = \langle 2,5 \rangle$.

we need $\Delta t \approx dT = \frac{dT}{dt} \Delta t$.

$\vec{u} = \frac{\nabla T(x(t^*), y(t^*))}{\|\nabla T(x(t^*), y(t^*))\|}$

What is $\frac{dT}{dt}$ in this direction?

$$\frac{dT}{dt} = \nabla T \cdot \vec{r}\,'(t) = \nabla T \cdot \vec{v}(t) = \nabla T \cdot \underbrace{\|\vec{v}(t)\|}_{\text{speed}} \underbrace{\vec{u}}_{\text{direction vector}}$$

$$\frac{dT}{dt} = \langle 2,5 \rangle \cdot \sqrt{5} \cdot \frac{\langle 2,5 \rangle}{\sqrt{4+25}} = \frac{\sqrt{5}}{\sqrt{29}}[4+25] = \frac{29}{\sqrt{29}}\sqrt{5}$$

$$= \frac{29\sqrt{29}}{29}\sqrt{5} = \sqrt{29}\sqrt{5}$$

$$\Rightarrow \frac{dT}{dt}\Big|_{t^*} \text{ in direction of } \nabla T|_{t^*} = \sqrt{29} \cdot \sqrt{5}$$

$$\Delta t \approx \sqrt{29}\sqrt{5}(0.1)$$

#5 $f(x,y) = \frac{y^3}{3} + \frac{x^3}{3}(1-y)$

Find linear and quadratic approximations around (1,1) and upper bound on error of linear approximation.

$$f(x,y) \approx \overbrace{f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}^{T_1(a,b) \Rightarrow z = T_1(a,b) \text{ is tangent plane}}$$

$$\underbrace{+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b) \right]}_{T_2(a,b)}$$

$z = T_1(a,b)$ Tplane with normal $\langle f_x(a,b), f_y(a,b), \pm 1 \rangle$

$L(x,y) = T_1(a,b)$ (linear approx ; aka tangent plane approx)

$Q(x,y) = T_1(a,b) + T_2(a,b)$ (quadratic approximation).

Error in $L(x,y)$: $f(x,y) = L(x,y) + E$

By Taylor's theorem, E on using $L(x,y)$ only is just the second term of Taylor series evaluated at some point (c,d) between (x,y) and (a,b).

Assuming we know $|x-a| \leq \alpha$ and $|y-b| \leq \beta$ and if $|f_{xx}|, |f_{yy}|, |f_{xy}| \leq M$ in this interval,

$$E \leq \frac{1}{2}\left[\alpha^2 M + 2\alpha\beta M + \beta^2 M \right] = \frac{1}{2}(\alpha+\beta)^2 M$$

In the case of $f(x,y) = \frac{y^3}{3} + \frac{x^3}{3}(1-y)$ and

$|x-1| \leq 0.2$ and $|y-1| \leq 0.2$

$\Rightarrow \quad 0.8 \leq x \leq 1.2 \quad$ and $\quad 0.8 \leq y \leq 1.2$

$$f_x = \frac{3x^2}{3}(1-y) = x^2(1-y) = x^2 - x^2y$$

$$f_{xx} = 2x(1-y)$$

$$f_{xy} = -x^2$$

$$f_y = \frac{3y^2}{3} - \frac{x^3}{3} = y^2 - \frac{x^3}{3} \quad ; \quad f_{yy} = 2y$$

$$|f_{xx}| = |2x| \cdot |1-y| \leq 2(1.2) \cdot (1-0.8) = 2(1.2)(0.2)$$

$$|f_{yy}| = |2y| \leq 2(1.2)$$

$$|f_{xy}| = x^2 \leq (1.2)^2$$

set $M = 2(1.2) = 2.4$, the largest of the three.

$$E \leq \frac{1}{2}M(\alpha+\beta)^2 = \frac{1}{2}(2.4)[0.2+0.2]^2 = \frac{1}{2}(2.4)(0.4)^2$$

$$= 1.2(0.16)$$

Ex) (Spring 2012), #3

Find the largest product of positive numbers x and y where $x+y^2=8$.

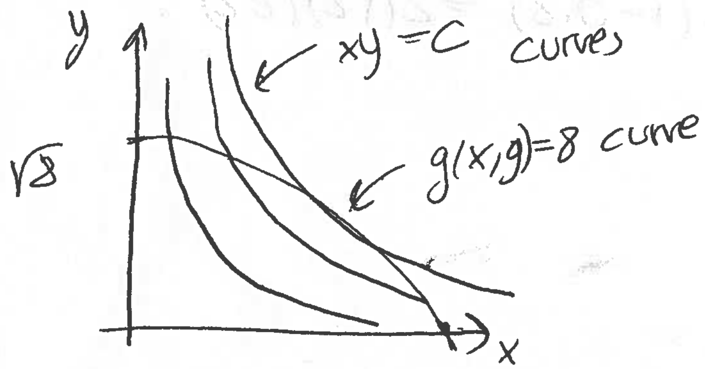
(a) Draw the constraint curve

$f(x,y) = xy$ (to be maximized)

$g(x,y) = x+y^2 = 8$ (constraint)

Notice that $y^2 = 8-x \Rightarrow$ parabola $\Rightarrow x = 8-y^2$

$xy = C$ are level curves



Graphical Interpretation of Lagrange Multiplier method

$\nabla f = \lambda \nabla g$ means the constraint curve is tangent to a level curve. There should be only one place where this occurs in this particular case.

$\nabla f = \langle y, x \rangle = \lambda \nabla g = \lambda \langle 1, 2y \rangle$

since $y>0$

$$\left.\begin{array}{l} y = \lambda \\ x = 2\lambda y = 2y^2 \end{array}\right\} \quad x+y^2=8 \Rightarrow 2y^2+y^2=8 = 3y^2=8 \Rightarrow y=\sqrt{\frac{8}{3}}$$

and $x = 8-y^2 = 8-\frac{8}{3} = \frac{16}{3}$

Ex) (Sping 2012), #1

$$f(u,v) = \frac{u^3}{3} + \frac{v^3}{3} - \frac{v^2}{2} - u + 2$$

(a) find critical points

$$\nabla f = \langle u^2-1, v^2-v \rangle = \langle 0,0 \rangle$$

$u^2=1 \Rightarrow u = \pm 1$

$v^2-v=0 \Rightarrow v(v-1)=0 \Rightarrow v=0$ or $v=1$

$\Rightarrow (1,0), (1,1), (-1,0), (-1,1).$

(b) classify them

$f_u = u^2-1 \Rightarrow f_{uu} = 2u$

$f_v = v^2-v \Rightarrow f_{vv} = 2v-1$

$f_{uv} = 0$

$$D = \begin{vmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{vmatrix} = f_{uu} f_{vv} - f_{uv}^2 = 2u(2v-1)$$

$D(-1,1) = -2 < 0 \Rightarrow$ saddlept

$D(-1,0) = 2$ and $f_{uu}(-1,0) = -2 < 0$

$\Downarrow$

local ma

$D(1,0) = -2 \Rightarrow$ saddle

$D(1,1) = 2$ and $f_{uu}(1,1) = 2 > 0 \Rightarrow (1,1)$ is local min

Suppose u is a function of α and β but v is only a function of β. Write down $\frac{\partial f}{\partial \beta}$.

$$\frac{\partial f}{\partial \beta} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial \beta} + \frac{\partial f}{\partial v}\frac{dv}{d\beta}$$

$u = u(\alpha, \beta)$ and $v = v(\beta)$.

$\frac{\partial f}{\partial u} = (u^2 - 1)$ and $\frac{\partial f}{\partial v} = (v^2 - v)$

$$\Rightarrow \frac{\partial f}{\partial \beta} = (u^2 - 1)\frac{\partial u}{\partial \beta} + (v^2 - v)\frac{dv}{d\beta}$$