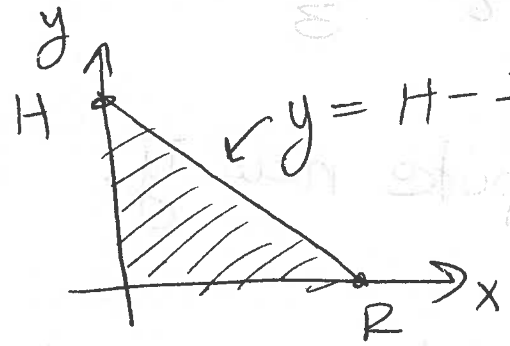
Review Sheet for Exam 3, Sergey Voronin ①

Center of mass in 2D

$$\bar{x} = \frac{M_y}{M} = \frac{1}{M}\iint_D x\rho(x,y)\,dA$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{M}\iint_D y\rho(x,y)\,dA$$

Ex)



find center of mass of this triangular shape $\rho(x,y) = \rho$ (const. density)

eq of line through $(R,0)$ and $(0,H)$:

$$y - y_1 = m(x - x_1) \Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{H-0}{0-R} = -\frac{H}{R}$$

$$y - H = -\frac{H}{R}(x-0) \Rightarrow y = H - \frac{H}{R}x$$

$$M = \iint_D \rho\,dA = \rho\iint_D dA = \rho A = \rho\underbrace{\left(\tfrac{1}{2}HR\right)}$$

because ρ = constant

comput area in simplest way if possible.

$$\bar{x} = \frac{1}{M}\iint_D x\rho(x,y)\,dA = \frac{\rho}{M}\iint_D x\,dA$$

$$\bar{x} = \frac{1}{\frac{1}{2}HR} \iint_D x\,dA = \frac{1}{\frac{1}{2}HR} \int_0^R \int_0^{H-\frac{H}{R}x} x\,dy\,dx$$

first bounds always numeric... how does vary once x fixe

$$= \frac{2}{HR} \int_0^R x\left(H - \frac{H}{R}x\right)dx$$

$$= \frac{2}{HR}\int_0^R Hx - \frac{H}{R}x^2\,dx = \frac{2}{HR}\left[\frac{HR^2}{2} - \frac{H}{R}\frac{R^3}{3}\right] =$$

$$= \frac{2}{HR} HR^2\left[\frac{1}{2} - \frac{1}{3}\right] = 2R\frac{1}{6} = \frac{R}{3}$$

so $\bar{x} = \frac{R}{3}$. Let us compute now $\bar{y}$.

$$\iint_D y\,dA = \int_0^R \int_0^{H-\frac{H}{R}x} y\,dy\,dx =$$

$$= \int_0^R \frac{\left(H - \frac{H}{R}x\right)^2}{2}dx = \frac{1}{2}\int_0^R \left(H^2 + \frac{H^2}{R^2}x^2 - 2\frac{H^2}{R}x\right)$$

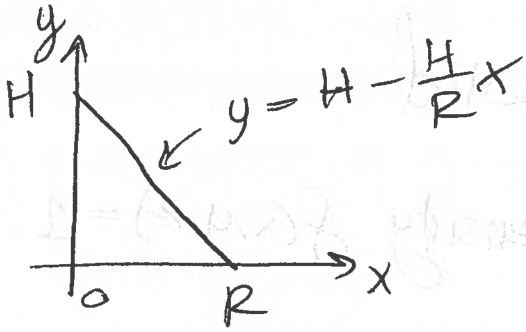
$$= \frac{1}{2}\left[H^2R + \frac{H^2}{R^2}\frac{R^3}{3} - 2\frac{H^2}{R}\frac{R^2}{2}\right] =$$

$$= \frac{1}{2}\left[H^2R + \frac{H^2}{3}R - H^2R\right] = \frac{1}{6}H^2R$$

$$\Rightarrow \bar{y} = \frac{1}{\frac{1}{2}HR}\frac{1}{6}H^2R = \frac{2}{HR}\frac{1}{6}H^2R = \frac{1}{3}H = \frac{H}{3}$$

$\Rightarrow$ so center of mass (centroid) is at $(\bar{x}, \bar{y}) = \left(\frac{R}{3}, \frac{H}{3}\right)$

Note in previous double integral how we picked the bounds.



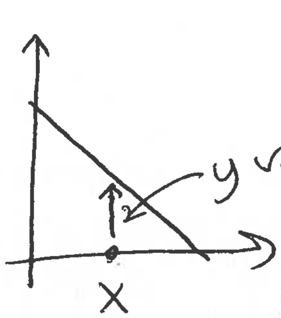
we have order dy, dx

$$\iint x\, dA = \iint x\, dy\, dx$$

step 1: how does x vary? must be numeric.

x goes from 0 to R

step 2: fix x (some value between 0 to R).
How does y vary?



y varies between y=0 and the line
line is a function of x

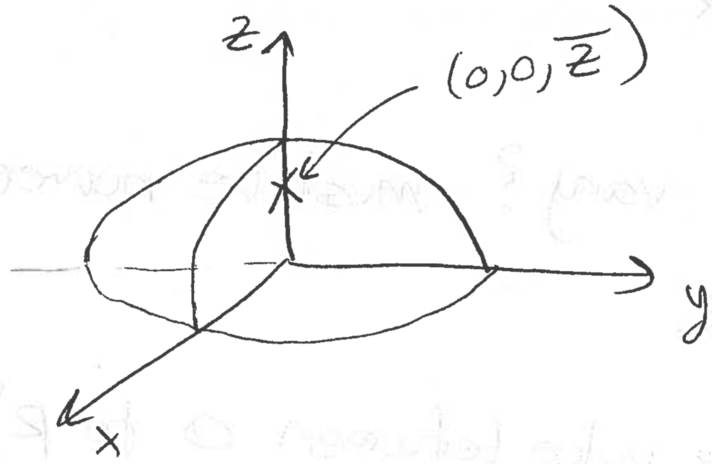
$$\Rightarrow \iint x\, dy\, dx = \int_0^R \int_0^{H-\frac{H}{R}x} x\, dy\, dx$$

Center of mass in 3D

Ex) Find center of mass of solid

$\{z \geq 0,\ x^2+y^2+z^2 \leq a^2\}$ if density $\delta(x,y,z) = 1$.

The solid is a half sphere.



First, look for symmetry! obviously, since density is constant and figure is symmetric around z-axis then center of mass must lie on z-axis.

$\Rightarrow$ center of mass $= (0,0,\bar{z})$

$$\bar{x} = \frac{M_{yz}}{M} \ ; \ \bar{y} = \frac{M_{xz}}{M} \ ; \ \bar{z} = \frac{M_{xy}}{M}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{1}{M} \iiint_S z\,\delta(x,y,z)\,dV = \frac{1}{M} \iiint_S z\,dV$$

since $\delta = 1$.

$$M = \iiint_S \delta(x,y,z)\,dV = \iiint_S 1\,dV = \text{Volume} \cdot \underset{\text{density}}{\underline{1}}$$

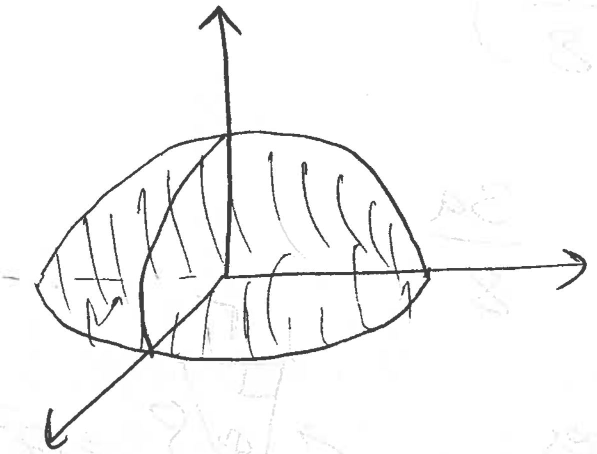
since $\delta = 1$. (In general you must integrate over density function

we know $V_{sphere} = \frac{4}{3}\pi a^3$ ⟶ must know basic facts...
$\quad$ ↳ radius a

$$\Rightarrow M = \frac{1}{2} V_{sphere} \cdot 1 = \frac{2}{3}\pi a^3$$

It remains to compute $M_{xy} = \iiint_S z\,dV$

easiest is to express region S in spherical coordinates



(ρ, ϕ, θ)

$0 \leq \theta \leq 2\pi$ (we go all way around)

$0 \leq \phi \leq \frac{\pi}{2}$ (we go only half way down)

$0 \leq \rho \leq a$ (all pts within half sphere including surface

$$\iiint_S z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \underbrace{(\rho\cos\phi)}_{z} \underbrace{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}_{\text{spherical coords volume element "}dV\text{"}}$$

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} (\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \underbrace{\sin\phi\cos\phi}_{\frac{1}{2}\sin(2\phi)} \left(\int_0^a \rho^3\, d\rho\right) d\phi\, d\theta$$

$$= \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi/2} \frac{1}{2}\sin(2\phi)\, d\phi\, d\theta$$

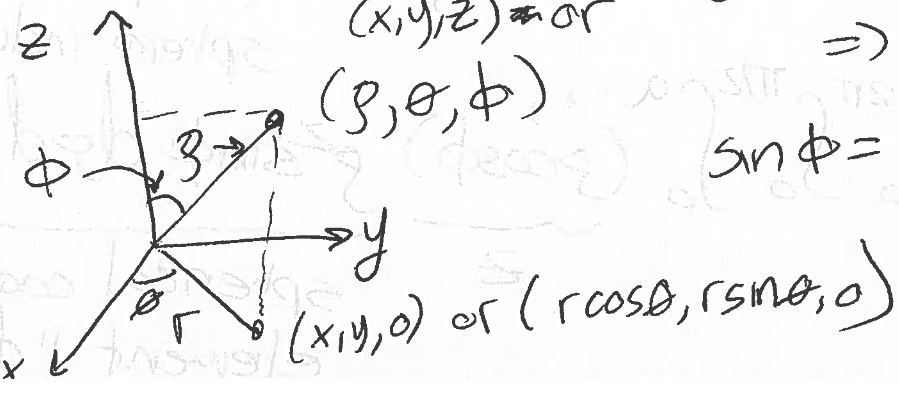
$$= -(-1-1) = 2$$

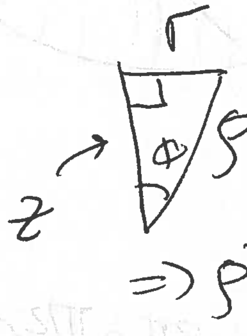
$$= \frac{a^4}{8}\int_0^{2\pi} \frac{-\cos(2\phi)}{2}\Bigg|_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta = 2\pi\,\frac{a^4}{8} = \pi\frac{a^4}{4}$$

$$\Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{3a}{8}\right)$$

$$\text{since } \bar{z} = \frac{\pi\frac{a^4}{4}}{\frac{2}{3}\pi a^3} = \frac{3a}{8}$$

Review spherical coordinates





$\Rightarrow z^2 + r^2 = \rho^2$

$\Rightarrow \rho^2 = x^2 + y^2 + z^2$

$\sin\phi = \frac{r}{\rho}$; $\cos\phi =$

Then,

$z = \rho\cos\phi$ and $r = \rho\sin\phi$

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$dV = \underbrace{\rho^2\sin\phi}\, d\rho\, d\phi\, d\theta$

this comes from change of variables determinant.

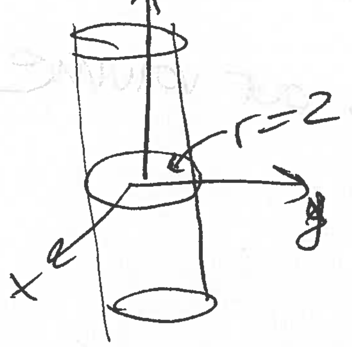
Ex) what is $\rho\sin\phi = 2$ $\Rightarrow$ spherical coordinates, 3D shape

$\Rightarrow \rho^2\sin^2\phi = 4 \Rightarrow \rho^2\sin^2\phi + \rho^2\cos^2\phi = 4 + \rho^2\cos^2\phi$

$\Rightarrow \rho^2(\sin^2\phi + \cos^2\phi) = 4 + \rho^2\cos^2\phi = 4 + z^2$

$\Rightarrow \rho^2 = x^2 + y^2 + z^2 = 4 + z^2 \Rightarrow \underbrace{x^2 + y^2 = 4}$

this is a cylinder, z can be any value.

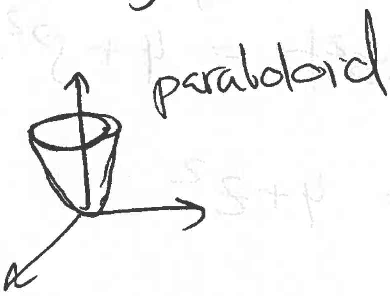


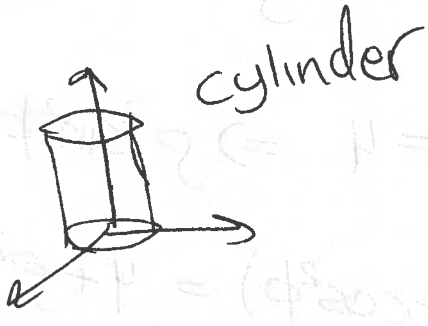
Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z)$

This is like drawing a plane at some fixed z and using polar coords in that plane.

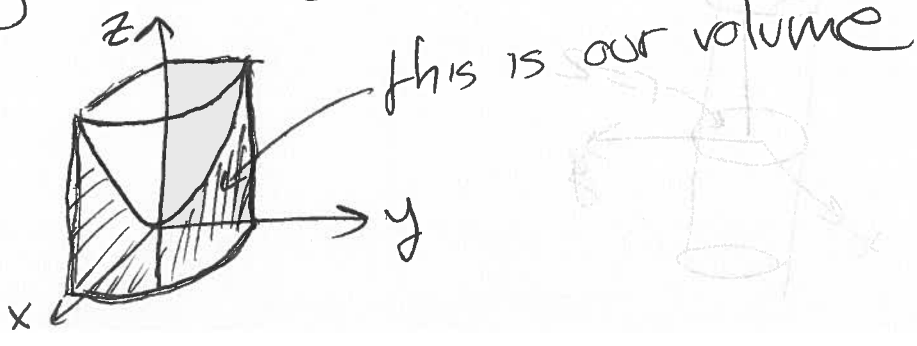
$$\iiint_E f(x,y,z)\,dV = \iiint f(r\cos\theta, r\sin\theta, z)\,\underbrace{r\,dz\,dr\,d\theta}_{\text{volume ele "dV"}}$$

Ex) Find the volume of the region above the xy-plane bounded by the paraboloid $z=x^2+y^2$ and the cylinder $x^2+y^2=a^2$.





exploit symmetry! Let us restrict to $x>0, y>0, z>$
This contains $\frac{1}{4}$th of total volume. It is $\frac{1}{4}$ since we do not go below xy-plane.



Let us describe region in cylindrical coordinates:

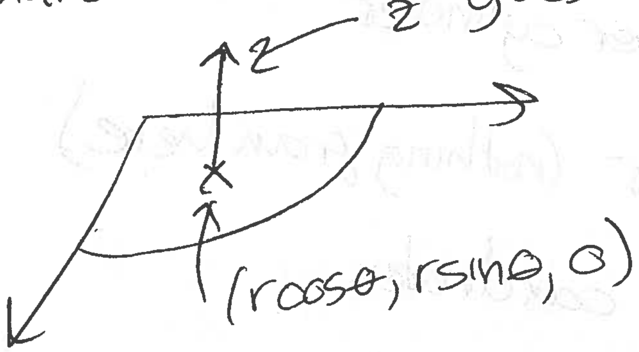
$0 \le \theta \le \frac{\pi}{2}$ (since we are in 1 "3D quadrant")

r goes from 0 to a. Why? project volume down:

$0 \le r \le a$

$0 \le \theta \le \frac{\pi}{2}$

How does z vary? Pick a point inside xy-plane above: z goes up to the paraboloid



$\Rightarrow 0 \le z \le \underbrace{x^2+y^2}_{z=x^2+y^2}$

$\Rightarrow 0 \le z \le r^2$ ← need z in terms of r, θ

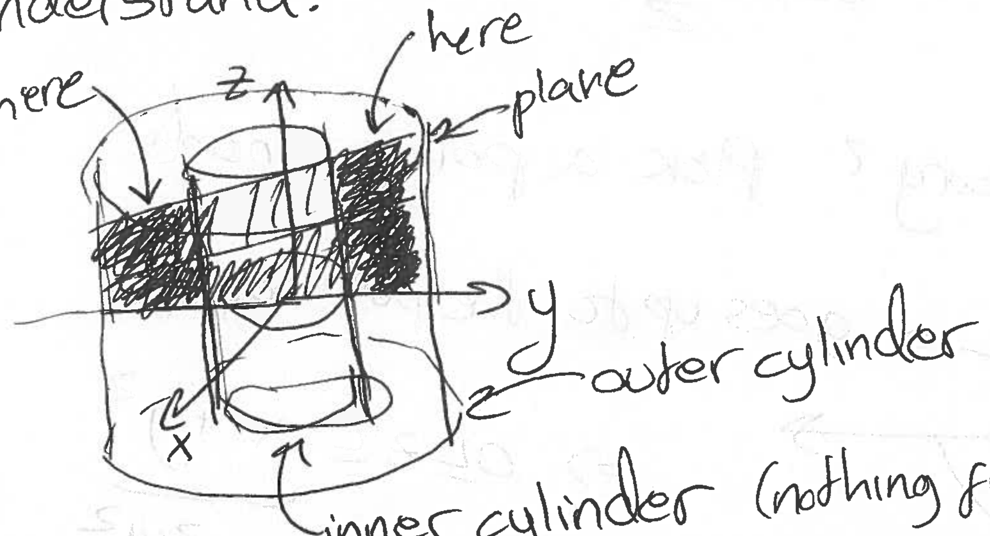
$$\Rightarrow V_{\frac{1}{4}} = \int_0^{\frac{\pi}{2}} \int_0^a \int_0^{r^2} 1 \cdot r \, dz \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^a r^3 \, dr \, d\theta = \int_0^{\pi/2} \frac{a^4}{4} d\theta = \frac{\pi}{2} \frac{a^4}{4}$$

$$\Rightarrow \text{Volume} = 4 \cdot V_{\frac{1}{4}} = \frac{\pi}{2} a^4$$

Ex) Evaluate $\iiint_E y\,dV$ where E is the region that lies below the plane $z=x+2$, above xy-plane ($z=0$) and between cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

$\Rightarrow$ Lots of constraints, make quick sketch to understand.



Describe region in cylindrical coordinates.

$1 \le r \le 2$ (between two cylinders)

$0 \le \theta \le 2\pi$ (we go all around z-axis).

$0 \le z \le \underbrace{x+2}_{r\cos\theta+2}$ (we go from xy plane $z=0$ up to plane $z=x+2$).

$$\Rightarrow \iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} \underbrace{(r\sin\theta)}_{y}\ \underbrace{r\,dz\,dr\,d\theta}_{\text{volume element}}$$

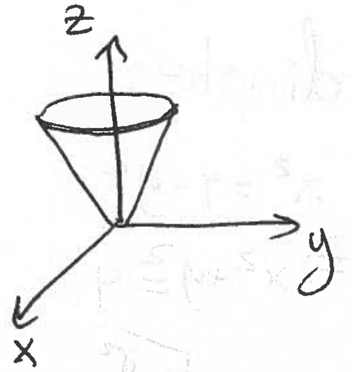
$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta\,(r\cos\theta+2)\,dr\,d\theta$$

$$= \cdots = \int_0^{2\pi} \frac{15}{8}\sin(2\theta) + \frac{14}{3}\sin(\theta)\,d\theta = 0.$$

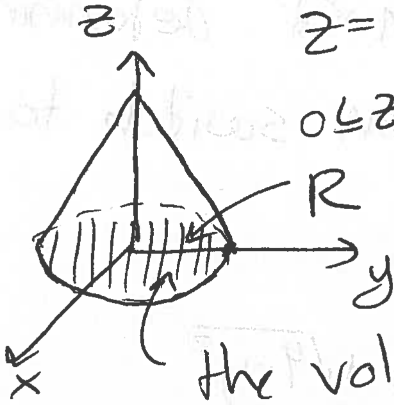
Polar coordinates

Sometimes, polar coordinates are sufficient (especially for double integrals).

Ex] Find volume V inside cone $z=\sqrt{x^2+y^2}$ for $0\le z\le 1$.



flip $\Longrightarrow$



$z = 1-\sqrt{x^2+y^2}$

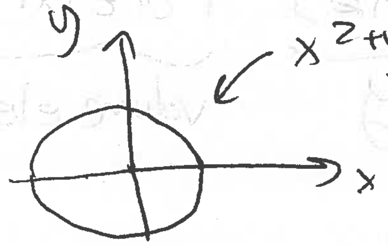
$0\le z\le 1$

R (circle)

the volume is the double integral over this area below the graph of $z=1-\sqrt{x^2+y^2}$

$$V = \iint_R \left(1-\sqrt{x^2+y^2}\right)dA$$

express R in polar:



$x^2+y^2=1 \Rightarrow \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{cases}$

$$V = \int_0^{2\pi}\int_0^1 \underbrace{(1-r)}_{\overset{\Uparrow}{1-\sqrt{x^2+y^2}}} \underbrace{r\,dr\,d\theta}_{\text{polar area element}}$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3} \Big|_0^1 \right) d\theta = 2\pi \cdot \frac{1}{6} = \frac{\pi}{3}$$

Ex) Evaluate $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2 \, dx\, dy$

Here we must determine what the integration region is, then switch to polar coordinates.

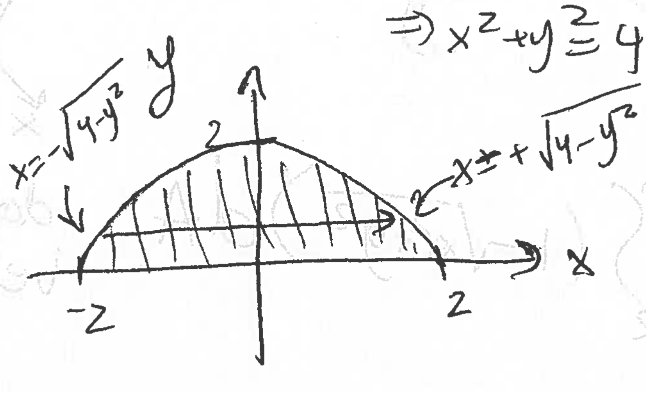
$x = \sqrt{4-y^2} \Rightarrow x^2 = 4-y^2$

$\Rightarrow x^2+y^2 = 4$

$0 \le y \le 2$

$-\sqrt{4-y^2} \le x \le +\sqrt{4-y^2}$

So this is a half disk:



So in polar coordinates, the region is simply:

$0 \le \theta \le \pi$ (not to 2π...)

$0 \le r \le 2$

$$I = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2 \, dx\,dy$$

$$= \int_0^{\pi} \int_0^2 \underbrace{(r\cos\theta)^2}_{x^2} \underbrace{(r\sin\theta)^2}_{y^2} \underbrace{r\,dr\,d\theta}_{\text{polar integration element}}$$

To evaluate this, must evaluate

$$\int \cos^2\theta \sin^2\theta \, d\theta$$

with $\cos^2\theta = \dfrac{1+\cancel{2}\cos(2\theta)}{2}$ and $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

See math calc III website for details. (review fall 2013)

Ex) Find the volume of the torus defined by the equation $\rho = \sin(\phi)$. Since this involves ρ, ~~[illegible]~~ this is in spherical coordinates.

~~[illegible]~~

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$0 \le \rho \le \sin\phi$

const density　　　volume element

$$\Rightarrow V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi} 1 \cdot \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{\text{volume element}}$$

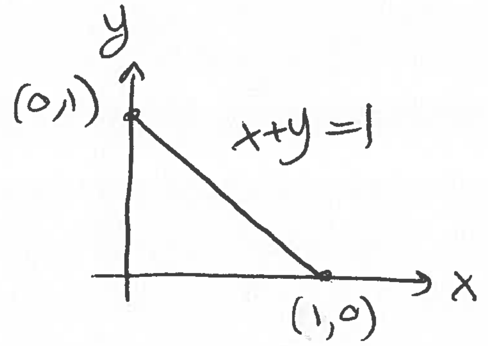
$$= \int_0^{2\pi}\int_0^{\pi} \left.\frac{\rho^3}{3}\right|_0^{\sin\phi} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\sin^4\phi}{3} \, d\phi \, d\theta$$

can integrate this via substitution $\sin^4\phi = \left[\frac{1-\cos(2\phi)}{2}\right.$

Let's now look at the example from HW, changing the order of integration. Notice that generally, having θ or ϕ on outside is ~~always~~ typically easier than having ρ or r on outside.

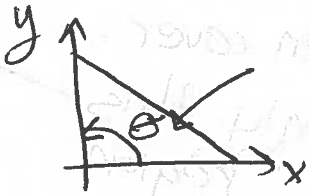
Ex (from HW) Find area.



First let's use polar coordinates with order dr, dθ

$$A = \iint_R r\,dr\,d\theta$$

Bounds on θ, the outermost integral must be numeric $\Rightarrow$



$0 \le \theta \le \frac{\pi}{2}$

Once we fix a θ, we vary r between 0 (at the origin) and the line $x+y=1$.

$$x+y = r\cos\theta + r\sin\theta = 1 \Rightarrow r = \frac{1}{\cos\theta+\sin\theta}$$

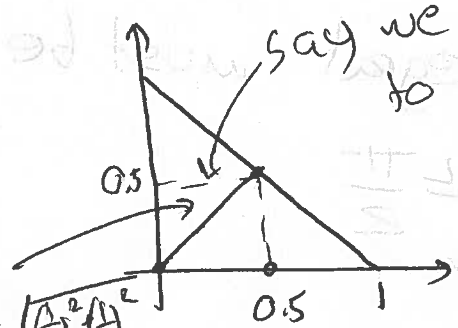
$$\Rightarrow A = \int_0^{\pi/2} \underbrace{\int_0^{\frac{1}{\cos\theta+\sin\theta}} r\,dr\,d\theta}_{\text{the result of this integral can depend only on } \theta}$$

otherwise A will not be numer

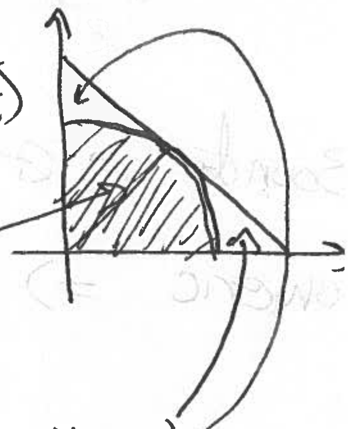
Let's now look at computing the area with the order $d\theta, dr$

$$A = \iint r \, d\theta \, dr$$

Bounds on r must be numeric. But note that we can't cover the whole region with a single set of bounds on r.



say we take r from 0 to $x+y=1$ line (with $0 \le \theta \le \frac{\pi}{2}$) but this will then cover: only this region.



these two regions remain (regions II and III).

$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$

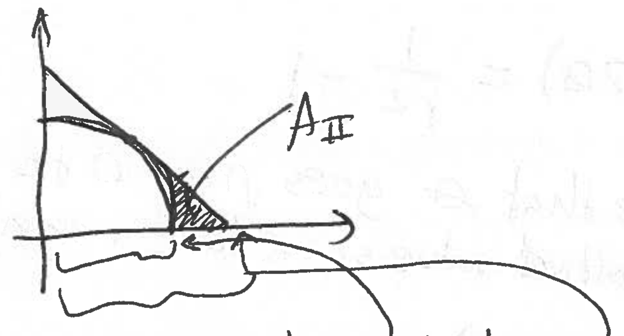
$A = A_{I} + A_{II} + A_{III}$ (region I)

$$A_{I} = \int_{r=0}^{r=\frac{1}{\sqrt{2}}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} r \, d\theta \, dr$$

This is just $\frac{1}{4}$ of area of circle of radius $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Note that by <u>symmetry</u>, $A_{II} = A_{III}$.

It remains to compute A_{II}.

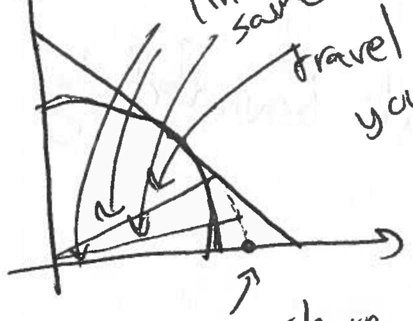


$$A_{II} = \iint r\,d\theta\,dr$$

First pick (find) the numeric bounds on r. r will range from the radius of the circle $\left(\frac{1}{\sqrt{2}}\right)$ to 1 (since $x+y=$ intersects $y=0$ at $x=1$).

$$\Rightarrow \frac{1}{\sqrt{2}} \le r \le 1$$

Having fixed a value of r, you travel from $y=0$ up to some value up to the line $x+y=1$. (by changing θ from 0 up to some value) how does θ vary?



$\Rightarrow$ θ varies from 0 until we are on $x+y=1$.

$\Rightarrow r\cos\theta + r\sin\theta = 1 \Rightarrow$ square both sides

$$r^2\left[\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta\right] = 1$$

$$\Rightarrow r^2\left(1+\sin(2\theta)\right)=1$$

$$\Rightarrow 1+\sin(2\theta)=\frac{1}{r^2} \quad \Rightarrow \sin(2\theta)=\frac{1}{r^2}-1$$

$$\theta=\frac{1}{2}\sin^{-1}\left(\frac{1}{r^2}-1\right)$$ (note that θ goes from 0 to a positive value so we take + sign.

$$\Rightarrow A_{II}=\int_{r=\frac{1}{\sqrt{2}}}^{r=1}\underbrace{\int_{\theta=0}^{\theta=\frac{1}{2}\sin^{-1}\left(\frac{1}{r^2}-1\right)} r\,d\theta\,dr}$$

note that result of inner integral can only depend on r for A_{II} to evaluate to a number.

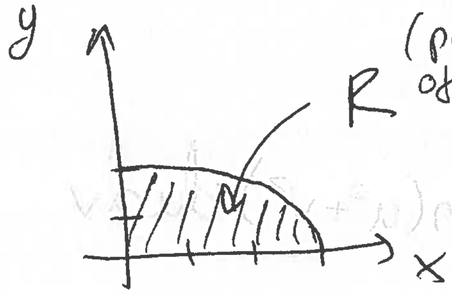
$A_{III}=A_{II}$.

Change of variables

Ex 12.8.26 $\iint_R \sin(9x^2+4y^2)\,dA$

where R is the region in 1st quadrant bounded by the ellipse $9x^2+4y^2=1$.

First quadrant means $x>0, y>0$.



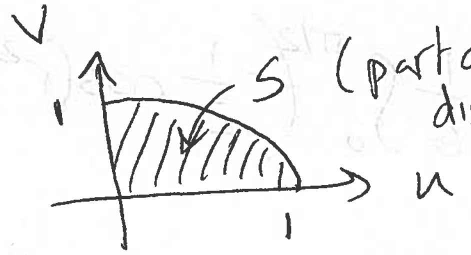
(part of ellipse) we can typically map elliptical regions to circular ones.

$\Rightarrow$ Let $u = 3x$ and $v = 2y \Rightarrow x>0, y>0 \Rightarrow u>0, v>$

and

$\Rightarrow u^2 + v^2 = 9x^2 + 4y^2 = 1$

$\Rightarrow$ using $u(x,y)$, $v(x,y)$ we map to



In this case easy to get inverse map by solving for $x(u,v)$ and $y(u,v)$.

$$\Rightarrow x(u,v) = \frac{u}{3} \; ; \; y(u,v) = \frac{v}{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \text{determinant of Jacobian matrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{6}$$

$$\Rightarrow \iint_R \underbrace{\sin(9x^2+4y^2)}_{f(x,y)} \, dA = \iint_S \underbrace{\sin(u^2+v^2)}_{f(x(u,v),\,y(u,v))} \underbrace{\left|\frac{1}{6}\right|}_{\text{absolute value of determinant}} du\,dv$$

The transformed region in polar coords is;

$0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq r \leq 1$

$$\Rightarrow \iint_R \sin(9x^2+4y^2)\,dA = \iint_S \frac{1}{6}\sin(u^2+v^2)\,du\,dv$$

$$= \int_0^{\pi/2}\int_0^1 \frac{1}{6}\sin(r^2)\,r\,dr\,d\theta =$$

$$= \frac{1}{6}\int_0^{\pi/2}\left(-\frac{1}{2}\cos(r^2)\Big|_0^1\right)d\theta = \frac{\pi}{24}(1-\cos(1))$$

Note the trick for the last HW, when it's hard

to get $\frac{\partial(x,y)}{\partial(u,v)}$, evaluate instead

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

However, then your determinant will be in terms of x,y. Plug into integral and simplify as much as possible, then replace all else in terms of u and v.

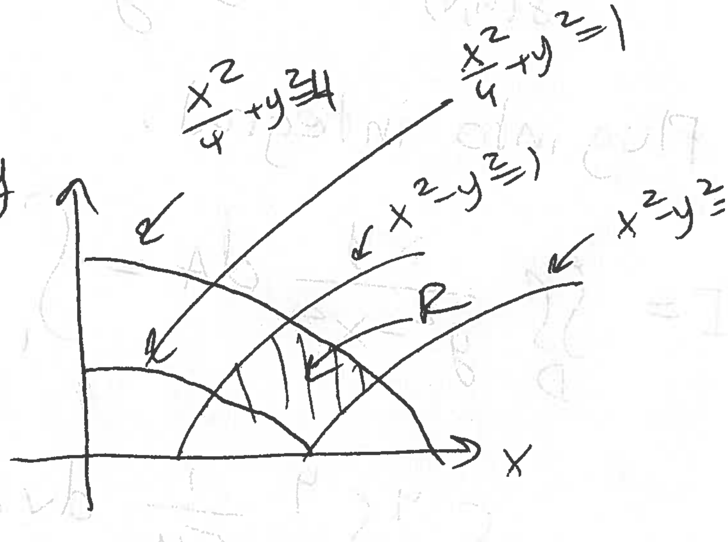
Ex from HW

$$I = \iint_D \frac{xy}{y^2-x^2}\, dA$$

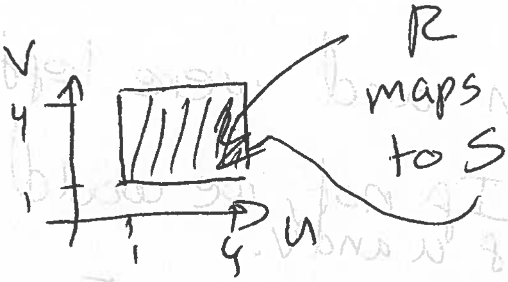
D is region in 1st quadrant $(x>0, y>0)$ bounded by $x^2-y^2=1$, $x^2-y^2=4$, and the ellipses $\frac{x^2}{4}+y^2=1$ and $\frac{x^2}{16}+\frac{y^2}{4}=1$.

Rewrite constraints:

$$\begin{cases} x>0,\ y>0 \\ x^2-y^2=1 \\ x^2-y^2=4 \\ \frac{x^2}{4}+y^2=1 \\ \frac{x^2}{4}+y^2=4 \end{cases} \Rightarrow$$



Let $u(x,y) = x^2-y^2$ and $v(x,y) = \frac{x^2}{4}+y^2$



$u=1,\ u=4$
$v=1,\ v=4$

Now let's compute $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -2x & 2y \\ \frac{1}{2}x & 2y \end{vmatrix}$$

$$= -4xy - xy = -5xy$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{5xy} \quad \text{(in terms of } x \text{ and } y\text{)}$$

Plug into integral:

$$I = \iint_D \frac{xy}{y^2-x^2}\,dA = \int_1^4 \int_1^4 \frac{xy}{-u} \left| \frac{1}{-5xy} \right| dv\,du$$

$$= -\int_1^4 \int_1^4 \frac{1}{5u}\,dv\,du = -\int_1^4 \frac{3}{5u}\,du$$

$$= -\frac{3}{5}\ln u \Big|_1^4 = -\frac{3}{5}\left(\ln(4) - \ln(1)\right) = -\frac{3}{5}\ln(4)$$

Notice in this case we plugged in and were left with $\frac{1}{5u}$ (all in terms of u,v). If not, we would need to substitute for x and y in terms of u and v. $u = x^2 - y^2$, $v = \frac{x^2}{4} + y^2 \Rightarrow u + v = +\frac{5}{4}x^2$ $\Rightarrow x^2 = \frac{4}{5}(v+u)$

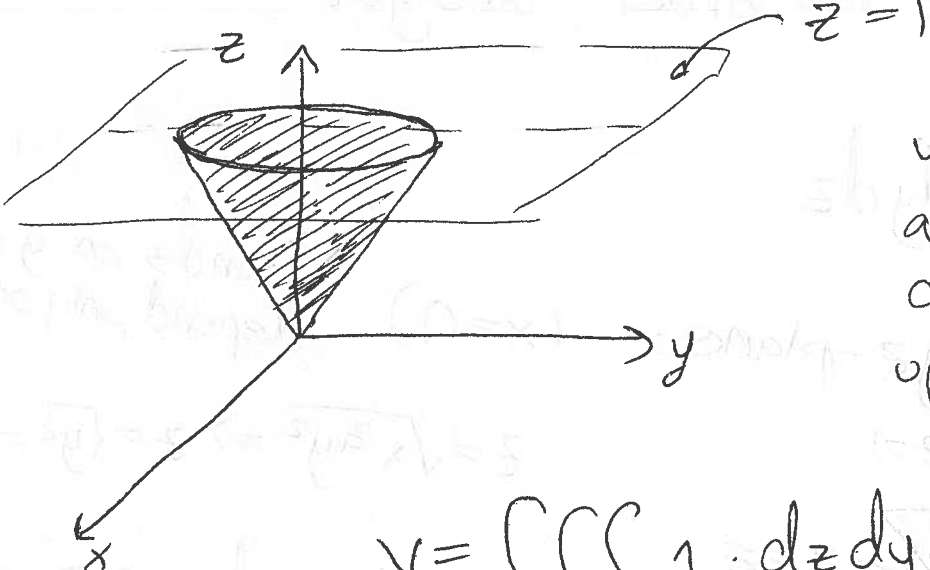
Using $x > 0$, can solve $x = \frac{2}{\sqrt{5}}(u+v)$, etc.

Ex) Consider finding volume of region

$$\sqrt{x^2+y^2} \le z \le 1$$

Let's write integrals in cartesian coordinates with order dz, dy, dx and dx, dy, dz.

$z = \sqrt{x^2+y^2}$ is a cone, $z=1$ is a plane

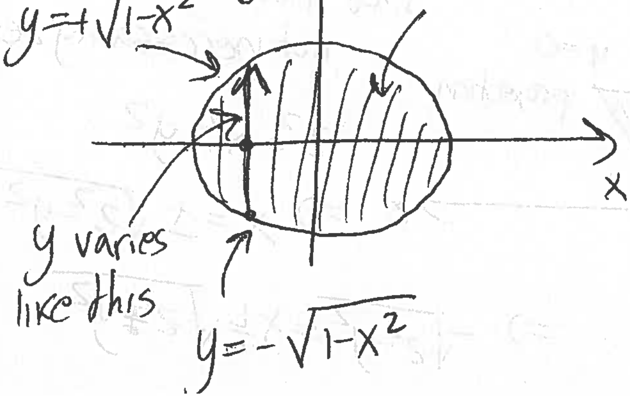


we are talking about the volume of this cone up to $z=1$.

$$V = \iiint 1 \cdot dz\,dy\,dx$$

To figure out bounds on x and y project onto the xy-plane:

disk $\sqrt{x^2+y^2} \le 1 \Rightarrow x^2+y^2 \le 1$



Bounds on x must be numeric: $-1 \le x \le 1$

Once x-fixed, y varies as:

$$-\sqrt{1-x^2} \le y \le +\sqrt{1-x^2}$$

$\sqrt{x^2+y^2} \le z \le 1$

$$\Rightarrow V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} 1 \cdot dz\, dy\, dx$$

numeric

can depend only on x

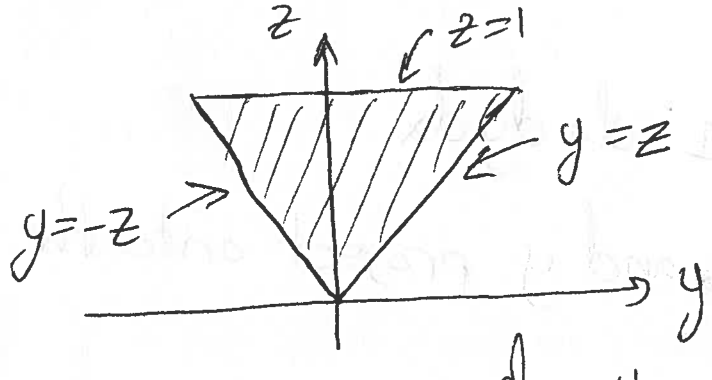
can depend on x and y

Let us now do the order $dx\,dy\,dz$:

$$V = \iiint dx\, dy\, dz$$

project onto yz-plane: $(x=0)$

bounds on y can depend only on z.



$z = \sqrt{x^2+y^2} \Rightarrow z = \sqrt{y^2} \Rightarrow z = $

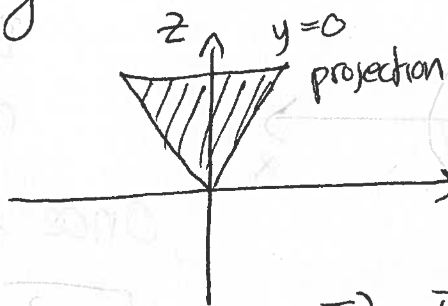
Now bounds on z must be numeric: $0 \le z \le 1$

Once z is fixed, y which can depend only on

max range of y with fixed z.

varies between $-z \le y \le +z$.

project onto xz-plane:



But now y not necessarily

$z^2 = x^2+y^2$

$\Rightarrow x = \pm\sqrt{z^2-}$

$\Rightarrow -\sqrt{z^2-y^2} \le x \le \sqrt{z^2-y^2}$

So $V = \int_0^1 \int_{-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} 1 \cdot dx\,dy\,dz$

numeric

can depend on z

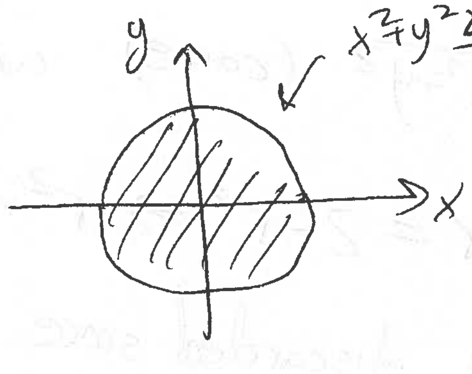
can depend on y and z.

Let's write the triple integral in cylindrical coordin[ates]

Let us use the order $dz, dr, d\theta$

$$V = \iiint \underbrace{r\,dz\,dr\,d\theta}_{\text{volume element}}$$

xy plane projection:



$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 1$

For z, $\underbrace{\sqrt{x^2+y^2}}_{r} \leq z \leq 1$

$$\Rightarrow V = \int_0^{2\pi} \int_0^1 \int_r^1 r\,\underbrace{dz\,dr\,d\theta}$$

note: often easiest to use this order for integration.

$$V = \int_0^{2\pi}\int_0^1\int_r^1 r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r(1-r)\,dr\,d\theta =$$

$$= \int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^3}{3}\right)\Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi}\frac{1}{6}\,d\theta = 2\pi\cdot\frac{1}{6} = \frac{\pi}{3}$$

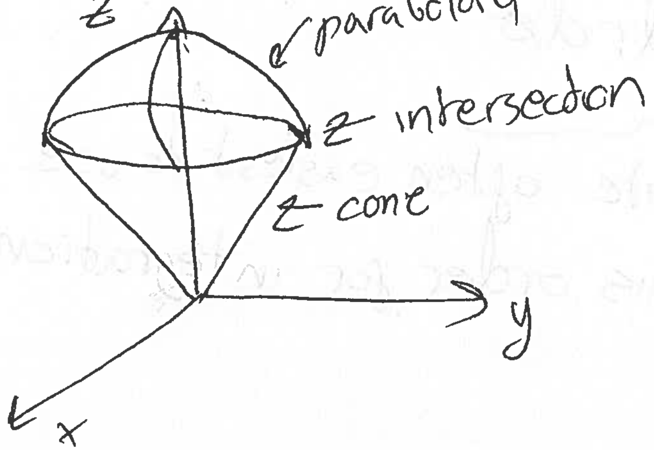
Note: on p. 6, we used polar coordinates to integrate $\iint_R (1-\sqrt{x^2+y^2})\,dA$ to get the same answer.

Ex) Consider ice-cream cone shape bounded by $z=\sqrt{x^2+y^2}$ (cone) and $z = 2-x^2-y^2$ (parabol

Intersection: $r = 2-r^2 \Rightarrow r^2+r-2=0 \Rightarrow (r-1)(r+2)=0$

$r=1$ ($r=-2$ discarded since $r\geq 0$).

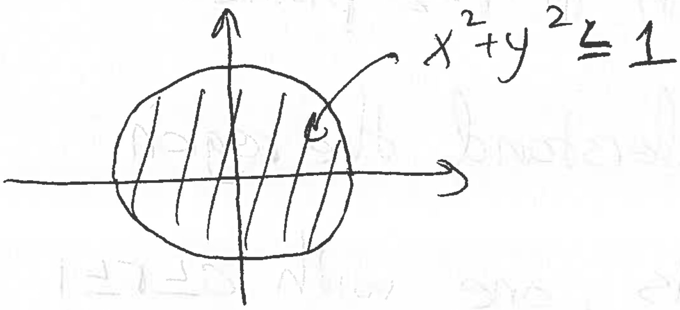
So the two intersect at $r=1$, forming circle $x^2+y^2=1$.



Let's use cylindrical coordinates

$\Rightarrow V = \iiint r\,dz\,dr\,d\theta$

To find bounds on r and θ, project onto xy-plane:



$\Rightarrow$ $0 \le \theta \le 2\pi$

$0 \le r \le 1$

z goes between cone and paraboloid (once θ, r are chosen).

$$\Rightarrow \underbrace{r}_{\sqrt{x^2+y^2}} \le z \le \underbrace{2-r^2}_{2-x^2-y^2}$$

$$\Rightarrow V = \int_0^{2\pi}\int_0^1\int_r^{2-r^2} r\,dz\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^1 r(2-r^2-r)\,dr\,d\theta = 2\pi\left(r^2 - \frac{r^3}{3} - \frac{r^4}{4}\right)\Big|_{r=0}^{r=1}$$

$$= 2\pi\left(1-\frac{1}{3}-\frac{1}{4}\right) = \frac{5\pi}{6}$$

Ex) (Exam 3, Spring 2014, #2)

$$V = \int_0^{2\pi}\int_0^1\int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{4-r^2}} r\,dz\,d$$

Based on volume element, we know this is in cylindrical coordinates.

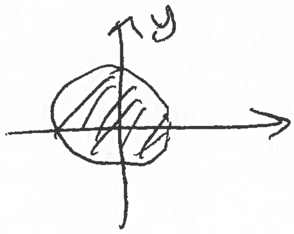
(a) Plot cross section in r-z plane.

First, we need to understand the region.

There are two integrals, one with $0 \le r \le 1$ and the other with $1 \le r \le 2$.

$x^2+y^2 \le 1$

$0 \le r \le 1 \Rightarrow$



$z=\sqrt{1-r^2} \Rightarrow x^2+y^2+z^2=1=1^2, \; z \ge 0$

$z=\sqrt{4-r^2} \Rightarrow x^2+y^2+z^2=4=2^2, \; z \ge 0$

when in this region, we are between spheres of radius 1 and 2 above xy-plane.

$1 \le x^2+y^2 \le 4$

$1 \le r \le 2 \Rightarrow$

when in this region of xy-plane, we are between $z=0$ and $x^2+y^2+z^2=2^2$, $z \ge 0$.

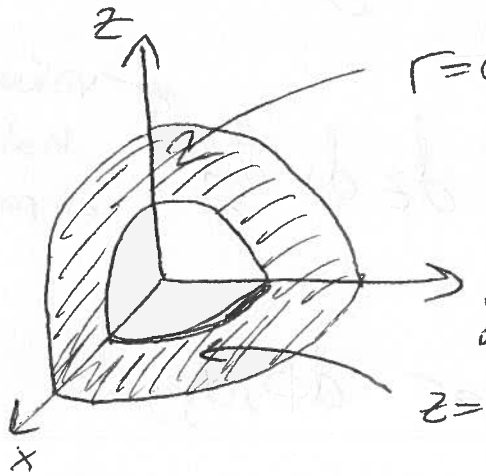
So the total volume is that inside sphere of radius 2 exterior to sphere of radius 1, for $z \ge 0$ (upper hemisphe

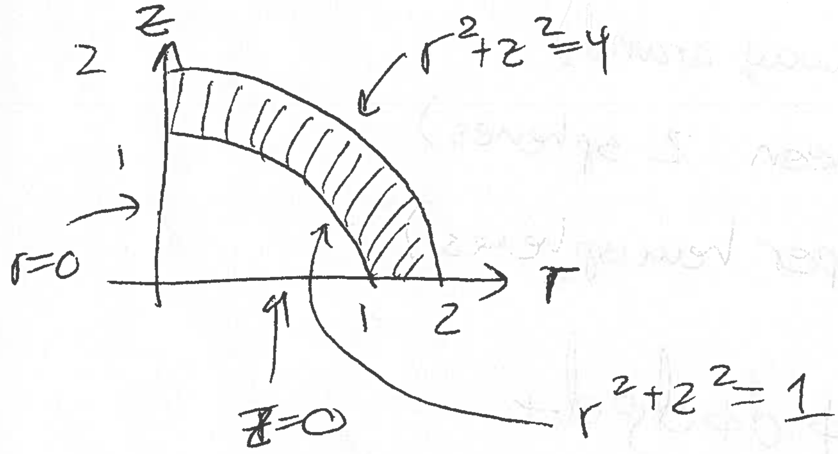
Sketch in rz-plane:

when $r=0$, z between 1 and 2.

when $z=0$, r between 1 and 2.

This is seen clearly from plot of region:

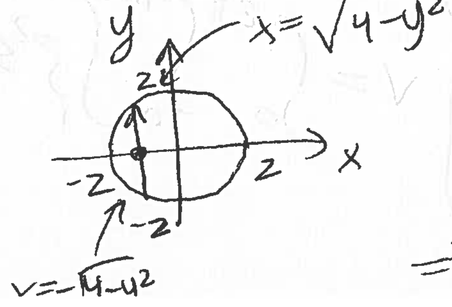




(b) write volume in cartesian as volume of sphere of radius 2 − volume of sphere of radius 1 (for $z \geq 0$).

Let's use order $dz\,dx\,dy$

rad 2:



y numeric $\Rightarrow -2 \leq y \leq 2$

$\Rightarrow -\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2}$

Similar for sphere of radius 1.

$$\Rightarrow V = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{0}^{\sqrt{4-y^2-x^2}} dz\,dx\,dy$$ ← volume of half sphere of radius 2

↑ dep on x and y $\Rightarrow z^2+x^2+y^2=2^2$

$$-\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\int_{0}^{\sqrt{1-y^2-x^2}} dz\,dx\,dy$$ ← volume of half sphere of radius 1.

(c) Write in spherical using order $d\phi, d\rho, d\theta$

easy $\Rightarrow$ $0 \le \theta \le 2\pi$ (all way around)

$1 \le \rho \le 2$ (between 2 spheres)

$0 \le \phi \le \frac{\pi}{2}$ (upper hemispheres)

$$\Rightarrow V = \int_{0}^{2\pi}\int_{1}^{2}\int_{0}^{\pi/2} \rho^2 \sin\phi \, d\phi\, d\rho\, d\theta$$

(d) use order $d\rho, d\phi, d\theta$

θ numeric $\Rightarrow$ $0 \le \theta \le 2\pi$

$\phi \Rightarrow$ indep of θ $\Rightarrow$ $0 \le \phi \le \frac{\pi}{2}$

$\rho \Rightarrow$ indep of θ, ϕ $\Rightarrow$ $1 \le \rho \le 2$

$$\Bigg\}\; V = \int_{0}^{2\pi}\int_{0}^{\pi/2}\int_{1}^{2} \rho^2 \sin\phi \, d\rho\, d$$

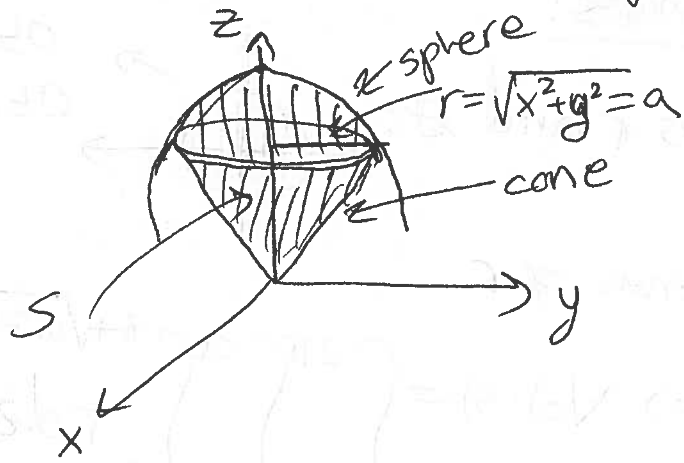
Ex) Ice Cream Cone Again

Consider the region bounded with

$$x^2+y^2+(z-a)^2=a^2 \quad \text{and} \quad z=\sqrt{x^2+y^2}$$

when $z=a \Rightarrow x^2+y^2=a^2$ and $a=\sqrt{x^2+y^2} \Rightarrow a^2=x^2+y^2$

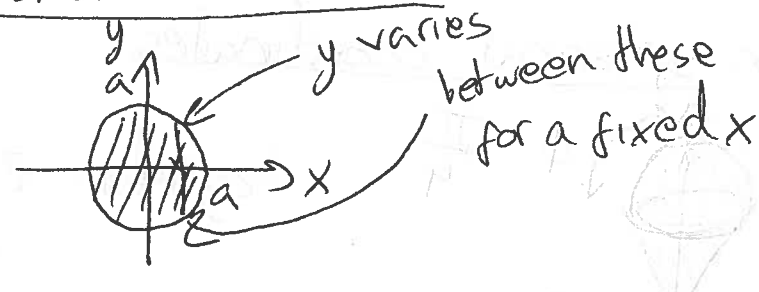
so the two surfaces match up:



bounded region S is the ice cream cone

(a) volume in cartesian coordinates:

project on xy-plane:



Let's use order dz dy dx $\Rightarrow$ x must be numeric

$\Rightarrow -a \le x \le a$. Once x is chosen, y varies as a function of x $\Rightarrow -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

z varies from cone to sphere:

$$\sqrt{x^2+y^2} \le z \le z \text{ on sphere}$$

$$x^2+y^2+z^2-2az+a^2=a^2 \Rightarrow x^2+y^2+z^2=2az$$

$$\Rightarrow z^2-2az+(x^2+y^2)=0$$

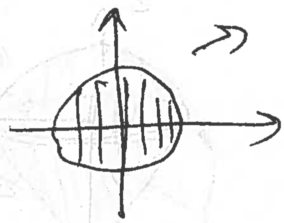
$$\Rightarrow z = \frac{2a \pm \sqrt{4a^2 - 4(x^2+y^2)}}{2} = a \pm \sqrt{a^2-(x^2+y^2)}$$

we pick the max $z \Rightarrow$ ~~a~~ $\underbrace{\sqrt{x^2+y^2}}_{\text{cone}} \le z \le \underbrace{a+\sqrt{a^2-(x^2+y^2)}}_{\text{sphere}}$

$$\Rightarrow \text{Volume}(S) = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{\sqrt{x^2+y^2}}^{a+\sqrt{a^2-(x^2+y^2)}} dz\,dy\,dx$$

(b) In cylindrical coordinates:

projection onto xy plane gives r and θ:

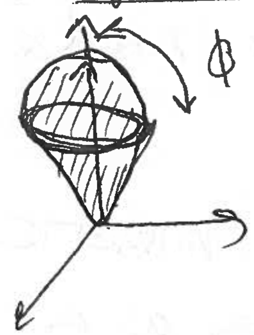


$\Rightarrow$ $0 \le r \le a$, $0 \le \theta \le 2\pi$

z same as in cartesian in terms of r:

$r \le z \le a+\sqrt{a^2-r^2}$ $\quad\Rightarrow \text{Vol}(S) = \int_0^{2\pi}\int_0^{a}\int_r^{a+\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

(c) In spherical coordinates:



$\phi = \frac{\pi}{4}$

cylinder: $z=\sqrt{x^2+y^2}$, set $x=0$

$\Rightarrow z=\pm y \Rightarrow 45°$ angle

$\Rightarrow 0 \le \phi \le \frac{\pi}{4}$ (up to boundary of cone)

Again, $0 \le \theta \le 2\pi$ (all way around)

$\rho = \sqrt{x^2+y^2+z^2}$ goes from 0 up to the sphere.

on sphere

$\Rightarrow x^2+y^2+z^2 = \rho^2 = 2az \Rightarrow \rho^2 = 2a\rho\cos\phi \Rightarrow \rho = 2a\cos\phi$

$$\text{Vol}(S) = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2a\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\pi/4} \frac{8a^3}{3}\cos^3\phi\sin\phi\,d\phi\right]d\theta = \ldots = \pi a^3$$