Final Review

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Differentials and Tangent Plane Approximations:

Eq of tangent plane to surface $z=f(x,y)$ at (x_0,y_0,z_0):

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Linear approximation:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

differential:

$$dz = f_x\,dx + f_y\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Ex) $f(x,y) = \exp(xy+y)$

If one moves from $(1,1)$ to $(1.1, 1.2)$, estimate the change in value of f. Take $dx = \Delta x = 0.1$; $dy = \Delta y = 0.2$

$$\Delta f \approx df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

$$= y e^{xy+y}\Big|_{(1,1)}(0.1) + (x+1)e^{xy+y}\Big|_{(1,1)}(0.2) = e^2(0.1) + 2e^2(0.2) = e^2/2$$

From (1,1), in what direction should one move to observe the greatest rate of increase in f?

$\Rightarrow$ in direction of gradient: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \Big|_{(1,1)} = \langle e^2, 2e^2$

What is $\frac{\partial f}{\partial s}$ in direction $\vec{u}$?

$\Rightarrow$ just the directional derivative: $\frac{\partial f}{\partial s} = \nabla f \cdot \vec{u}$

note: $\vec{u}$ is unit vector. For ex, if direction $\vec{A} = 3\hat{i} + 4\hat{j}$

then $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$. $\Rightarrow \frac{\partial f}{\partial s} = \langle e^2, 2e^2 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{3}{5}e^2 + \frac{8}{5}$

$= \frac{11}{5}e^2$

If one moves away from (1,1) in the direction of $\vec{A}$ at the constant speed $|\vec{v}| = v_0$ what is $\frac{df}{dt}$?

$$\frac{df}{dt} = \frac{df/ds}{dt/ds} = \left(\frac{1}{v_0}\right)^{-1} \frac{df}{ds} = \underbrace{v_0}_{(ds/dt)} \frac{df}{ds} = \frac{11}{5}e^2 v_0$$

Summer 2014 exam

Ex) Franklin the fish is floating through shark infested waters along the path $\vec{r}(t) = \langle t, 2-3t^2, \sin(t) \rangle$. The density of sharks in the water is given by $f(x,y,z) = e^{\cos(x)} + yz$

(a) At $t=\pi$, estimate the change in shark density Δf, if Franklin were to float for an <u>additional</u> <u>time</u> of $\Delta t = 0.1$.

$$\Rightarrow \Delta f \approx df = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \cdot \langle dx, dy \rangle =$$

$$= \nabla f \cdot \langle dx, dy \rangle = \nabla f \cdot \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle dt = \nabla f \cdot \vec{r}\,'(t)\, dt$$

$$\Rightarrow \Delta f \approx \nabla f \big|_{\vec{r}(t)} \cdot \vec{r}\,'(t) \cdot \Delta t$$

$$= \underbrace{\langle -\sin t\, e^{\cos t}, \sin t, 2-3t^2 \rangle} \cdot \langle 1, -6t, \cos t \rangle\, \Delta t$$

since $\nabla f = \langle -\sin x\, e^{\cos x}, z, y \rangle$, $\vec{r}\,'(t) = \langle 1, -6t, \cos t \rangle$
and $\vec{r}(t) = \langle t, 2-3t^2, \sin t \rangle$

$$= \langle 0, 0, 2-3\pi^2 \rangle \cdot \langle 1, -6\pi, -1 \rangle\, 0.1 = \frac{3\pi^2 - 2}{10}$$

(b) At $t=\pi$, estimate the change in shark density Δf, if Franklin were to float for an <u>additional distance</u> of $\Delta s = 0.1$.

Again, $\Delta f \approx df = \nabla f \cdot \vec{r}\,'(t)\, dt = \nabla f \cdot \vec{r}\,'(t) \underbrace{\frac{dt}{ds} ds}_{\text{notice how we introduce } ds}$

But $\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{\|\vec{r}'(t)\|}$

use $\underset{\sim}{ds} = \Delta s$

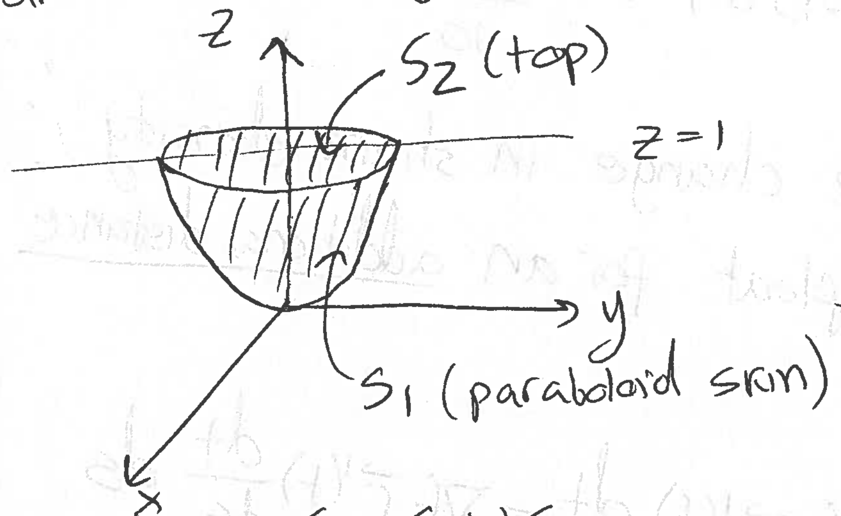
$\Rightarrow \Delta f \approx df = \nabla f\big|_{\vec{r}(t)} \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \, ds$

$= \langle 0, 0, 2-3\pi^2\rangle \cdot \langle 1, -6\pi, -1\rangle \cdot \frac{1}{\sqrt{2+36\pi^2}} (0.1)$

$= \frac{3\pi^2 - 2}{10\sqrt{2+36\pi^2}}$

Ex) <u>Flux</u>

Let E be the solid bounded below by $z = x^2 + y^2$ and above by $z = 1$. Let S be the surface of E and let $\vec{F} = \langle y+z, x+z, x+y\rangle$.



$S = S_1 \cup S_2$

(a) calculate the total outward flux.

flux $= \iint_{S_1} \vec{F} \cdot \vec{n} \, ds + \iint_{S_2} \vec{F} \cdot \vec{n} \, ds$

S_1: the surface $z = x^2 + y^2$ for $0 \le z \le 1$.

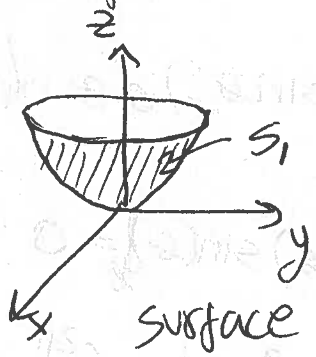
S_2: the surface $z = 1$ for $x^2 + y^2 \le 1$.

$$\text{flux}_{S_1} = \iint_{S_1} \vec{F} \cdot \hat{n}\, dS = \iint_{R_1} \vec{F} \cdot \frac{\pm \nabla g}{\|\nabla g\|} \cdot \frac{\|\nabla g\|}{|\nabla g \cdot \hat{p}|}\, dA$$

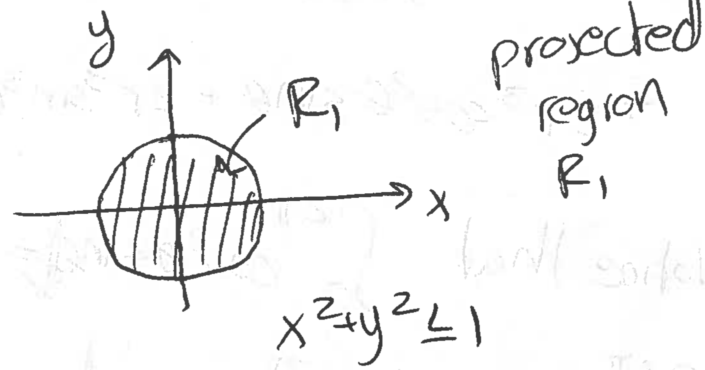
R_1 ↑ projected region

$$= \iint_{R} \vec{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot \hat{p}|}\, dA$$

We will project the surface S_1 onto the xy-plane:



$\Rightarrow$



or $0 \le r \le 1$; $0 \le \theta \le 2\pi$

$g(x,y,z) = x^2 + y^2 - z = 0$

$\Rightarrow \nabla g = \langle 2x, 2y, -1 \rangle$; $|\nabla g \cdot \hat{p}| = |\nabla g \cdot \hat{k}| = |-1| = 1$

$$\text{flux}_{S_1} = \iint_{R_1} \langle y+z, x+z, x+y \rangle \cdot \frac{\langle 2x, 2y, -1 \rangle}{1}\, dA$$

must replace all z in terms of x and y since R_1 is in xy-plane.

$$\Rightarrow \text{flux}_{S_1} = \iint_{R_1} \langle y+x^2+y^2,\ x+x^2+y^2,\ x+y\rangle \cdot \frac{\langle 2x, 2y, -1\rangle}{1}$$

$$= \iint_{R_1} (2xy + 2x^3 + 3xy^2 + 2xy + 2x^2y + 2y^3 - x - y)\, dA$$

Now switch to polar coordinates:

$x = r\cos\theta,\ y = r\sin\theta$

$$\Rightarrow \text{flux}_{S_1} = \int_0^1 \int_0^{2\pi} (4r^2\cos\theta\sin\theta + 2r^3\cos^3\theta + 3r^3\cos\theta\sin^2\theta$$

$$+ 2r^3\cos^2\theta\sin\theta + 2r^3\sin^3\theta - r\cos\theta - r\sin\theta)\, d\theta\, dr$$

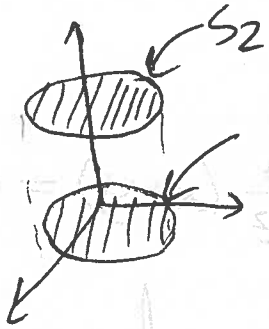
Notice that $\int_0^{2\pi} \cos^2\theta\sin\theta\, d\theta = 0$; $\int_0^{2\pi} \cos(\theta)\sin(\theta)\, d\theta = 0$, etc.

$\int_0^{2\pi} \cos(\theta)\, d\theta = \int_0^{2\pi} \sin(\theta)\, d\theta = 0$. $\int_0^{2\pi} \sin^3(\theta)\, d\theta = \int_0^{2\pi} \cos^3(\theta)\, d\theta = 0$

$\Rightarrow$ Notice these trig simplifications and how we choose to do the θ integral first! (see last page of this packet)

$\Rightarrow$ all terms are zero so $\text{flux}_{S_1} = 0$.

Now for the top surface S_2: $z=1$, $x^2+y^2 \leq 1$



R_2 (projection onto xy-plane)

note: S_2 lies in plane $z=1 \Rightarrow g(x,y,z)=z=1$

$\Rightarrow \nabla g = \langle 0,0,1\rangle$; $p=\hat{k} \Rightarrow |\nabla g\cdot \hat{p}|=1$ and $\|\nabla g\|=1$

$$\Rightarrow \text{flux}_{S_2} = \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{R_2} \vec{F}\cdot \frac{\pm\nabla g}{|\nabla g\cdot\hat{p}|}\,dA$$

$$= \iint_{R_2} \langle y+z,\ x+z,\ x+y\rangle \cdot \frac{\langle 0,0,1\rangle}{1}\,dA$$

$$= \iint_{R_2} (x+y)\,dA = \int_0^1 \int_0^{2\pi} (r\cos\theta + r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^1 0\,dr = 0.$$

again trig integrals from 0 to 2π these are zero.

$\Rightarrow$ total flux $= 0$!

(b) Note that the flux is in this case much easier to evaluate using the <u>divergence theorem.</u>

$$\Rightarrow \text{flux} = \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_E \nabla\cdot\vec{F}\,dV$$

(S bounds solid E
E is the filled paraboloid with top cover)

Now, $\vec{F} = \langle y+z, x+z, x+y \rangle$

$$\Rightarrow \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(x+z) + \frac{\partial}{\partial z}(x+y) = 0$$

$$\Rightarrow \text{total flux} = \iiint_E \nabla \cdot \vec{F} \, dV = \iiint_E 0 \, dV = 0 \, !$$

Ex) Vector fields and stokes theorem.
(#1, summer 2014).

$\vec{F} = \langle y, x+e^z, ye^z \rangle$

Surface S given by: $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{4}\right)^2 = 1$

for $z \geq 0$.

(a)
compute the line integral $\oint_C \vec{F} \cdot d\vec{r}$ where the path C is the boundary of the surface S.

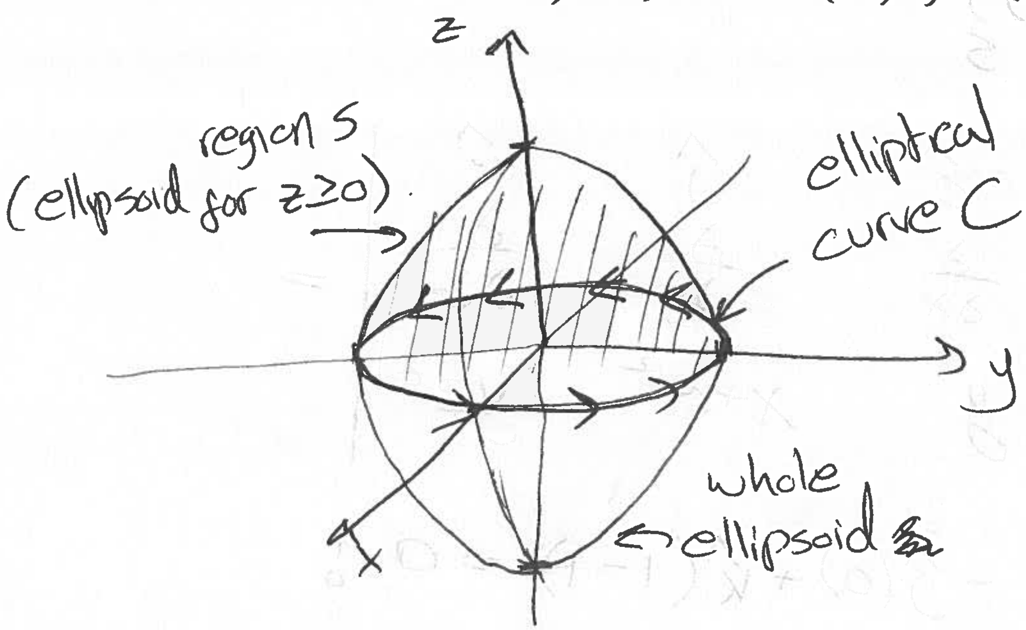
$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

Note: C is the ellipse path which bounds the surface of S.

S is the ellipsoid.

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{4}\right)^2 = 1$$

$\Rightarrow (\pm 2, 0, 0)$, $(0, \pm 3, 0)$, $(0, 0, \pm 4)$



which bounds S (as you go along C in counterclockwise direction, surface S is on your left)

$$C: \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \quad \Rightarrow \vec{r}\,'(t) = \langle -2\sin t, 3\cos t, 0\rangle$$

$$\vec{r}(t) = \langle 2\cos t, 3\sin t, 0\rangle \; ; \; 0 \le t \le 2\pi$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt =$$

$$= \int_0^{2\pi} \langle 3\sin t, 2\cos t + e^0, 3(\sin t)e^0 \rangle \cdot \langle -2\sin t, 3\cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi} -6\sin^2 t + 6\cos^2 t + 3\cos t \, dt$$

$= 0 \quad \Rightarrow$ again recall $\int_0^{2\pi}$ (trig functions) $= 0$ (for periodic func with period of multiples of π

(b) Can also use Stokes thm:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

Now $\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x+e^z & ye^z \end{vmatrix} =$

$$= \hat{\imath}(e^z - e^z) - \hat{\jmath}(0) + \hat{k}(1-1) = 0 \ !$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_S (0\cdot\hat{n})\,ds = 0!$$

(c) $\oint_C \vec{F}\cdot d\vec{r} = \oint_C \nabla f\cdot dr = f(\vec{r}(2\pi)) - f(\vec{r}(0))$

where f is the potential function of $\vec{F}$. That is,

$\vec{F} = \nabla f$. Find f by integrating.

$$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k} = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

$$= y\hat{\imath} + (x+e^z)\hat{\jmath} + ye^z\hat{k}$$

$$F_1 = y = \frac{\partial f}{\partial x} \Rightarrow f = xy + C_1(y,z)$$

$$F_2 = x + e^z \Rightarrow \frac{\partial f}{\partial y} = x + e^z \Rightarrow f = xy + e^z y + C_2(x,z)$$

$$F_3 = ye^z \Rightarrow \frac{\partial f}{\partial z} = ye^z \Rightarrow f = ye^z + C_3(x,y)$$

So setting ~~[illegible]~~ $f(x,y,z) = xy + e^z y$ we have $\vec{F} = \nabla f$. Note that f is not unique but the above choice works.

$\Rightarrow \vec{F}$ is conservative (we know also from $\nabla \times \vec{F} = 0$!)

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$

In fact, we can check:

$$\oint_C \vec{F} \cdot dr = f(\vec{r}(2\pi)) - f(\vec{r}(0)) = f(2,0,0) - f(2,0,0) = 0$$

$$= \int_{0}^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_C \nabla f \cdot dr = f(\underbrace{r(2\pi)}_{r(b)}) - f(\underbrace{r(0)}_{r(a)})$$

by Fundamental Theorem of Line Integrals.

Ex] (Lagrange Multipliers and Optimization, Summer 2014 exam)

You are walking along an elliptical path

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

pain function in this region is:

$$f(x,y) = 25x^2 + 9y^2 - x$$

ⓐ Use Lagrange multipliers to find min/max pain levels on your path.

to max/min: $f(x,y) = 25x^2 + 9y^2 - x$

constraint: $g(x,y) = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$

$\nabla f = \lambda \nabla g \Rightarrow \langle 50x - 1, 18y \rangle$; $\lambda \nabla g = \lambda \langle \frac{2}{9}x, \frac{2}{25}y \rangle$

$$\Rightarrow \begin{cases} 50x - 1 = \frac{2}{9}\lambda x \\ 18y = \frac{2}{25}\lambda y \\ \left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1 \end{cases}$$

← note constraint equation

Look at:

$$18y = \frac{2}{25}\lambda y$$

There are two possibilities:

① $y=0$

② $y \neq 0 \Rightarrow$ then can divide through by y and get $\lambda = \frac{25}{2}\cdot 18 = 25\cdot 9$

Plug into first equation:

$$50x - 1 = \frac{2}{9}(25\cdot 9)\cdot x = 50x$$

$\Rightarrow 0 \neq 1 \Rightarrow$ condition ② leads to a contradictio

Hence, we must have $y=0$! From the constraint equation this implies:

$$\left(\frac{x}{3}\right)^2 + 0 = 1 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

So the constrained points we obtain are:

$(-3,0)$ and $(3,0)$ $\Rightarrow$ we plug these into f.

$f(-3,0) = 25.9 + 3 \quad (\text{max})$

$f(3,0) = 25.9 - 3 \quad (\text{min})$

(b) Solve the optimization problem a different way.

The elliptical path $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$

can be parametrized with $\vec{r}(t) = 3\cos t\,\hat{i} + 5\sin t\,\hat{j}$; $0 \le t \le$

Then we can obtain a pain function of t along this pad

$$\bar{f}(t) = f(x(t), y(t)) = 25.9\cos^2 t + 9.25\sin^2 t - 3\cos t$$

$$= 25.9 - 3\cos t$$

Now we can use 1-variable calculus:

$\bar{f}'(t) = 3\sin t = 0 \Rightarrow t = 0, \pi,$ etc

$\bar{f}''(t) = 3\cos(t)$; $\bar{f}''(0) = 3 > 0$ (min)

$\bar{f}''(\pi) = -3 < 0$ (max)

$\bar{f}(\pi) = 25.9 + 3$ (max)

$\bar{f}(0) = 25.9 - 3$ (min)

Note the following calculus result:

$$\int_0^{2\pi} \cos(t)\,dt = \int_0^{2\pi} \sin(t)\,dt = \int_0^{2\pi} \sin^3(t)\,dt =$$

$$= \int_0^{2\pi} \cos^3(t)\,dt = \int_0^{2\pi} \cos^2(t)\sin(t)\,dt = \int_0^{2\pi} \sin^2(t)\cos(t)\,dt$$

$$= \int_0^{2\pi} \sin(t)\cos(t)\,dt = 0$$

The integrals above are all zero since the integrands have completed one or more periods from 0 to 2π (integral number). These are useful to recognize because often in polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) you get integrals like above.

Note that $\int_0^{2\pi} \cos^2(t)\,dt = \int_0^{2\pi} \sin^2(t)\,dt = \pi$.

From spring 2013 exam

Power series first two terms:

$$f(x,y) = f(x_0,y_0) + (x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)$$

$$+ \frac{1}{2!}\left[(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\right] + \cdots$$

error associated with using only two terms:

$$|E(x,y)| \le \frac{M}{3!}\left(|x-x_0| + |y-y_0|\right)^3$$

where $M \ge \{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\}$

for $|x-x_0| \le \alpha$ and $|y-y_0| \le \beta$. Look at example 2, spring 2013.

Ex) (spring 2013, #4)

$$I = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma \overset{\text{stokes theorem}}{=} \oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \vec{T}\, ds$$

But also, if you compute $\vec{G} = \nabla \times \vec{F}$, then can evaluate I using:

$$\iint_S \vec{G} \cdot \vec{n}\, d\sigma \underset{\text{Divergence thm}}{=} \iiint_E \nabla \cdot \vec{G}\, dV$$

Ex) (Another Flux Example)

Calculate the flux through the surface

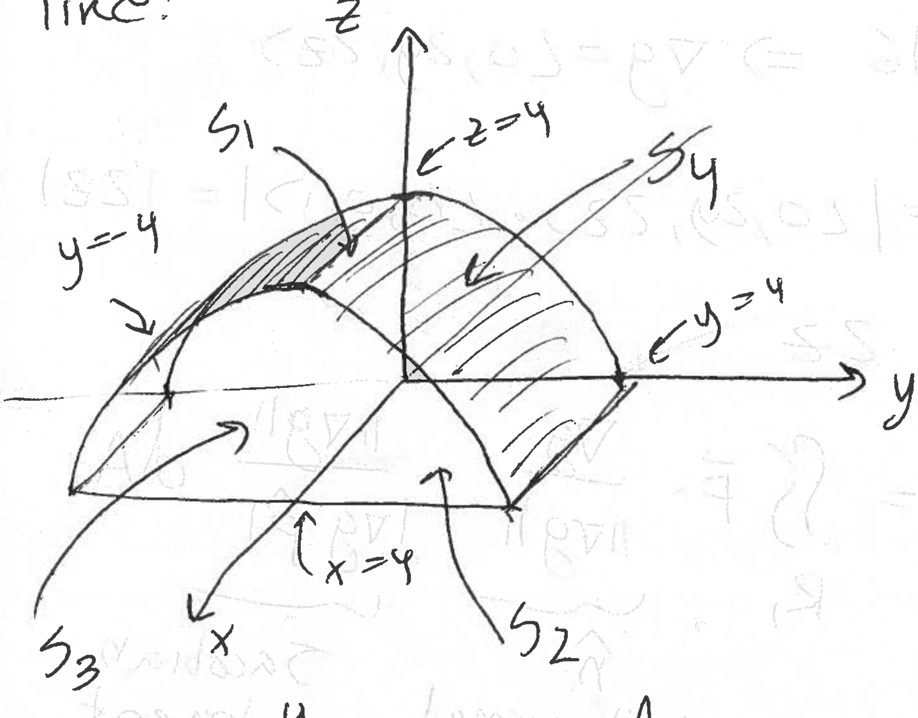
$0 \le z \le \sqrt{16-y^2}$, $0 \le x \le 4$

of vector field $\vec{F}(x,y,z) = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$

Let's plot: z is between $z=0$ and $z^2 = 16-y^2$

$z \le \sqrt{16-y^2} \Rightarrow -4 \le y \le 4$ and $0 \le x \le 4$ is given

$z^2+y^2=16$ is a paraboloid. So the shape looks like:



Notice this shape has 4 sides.

S_1 : top (curved)

S_2 : bottom (flat) part of xy-plane

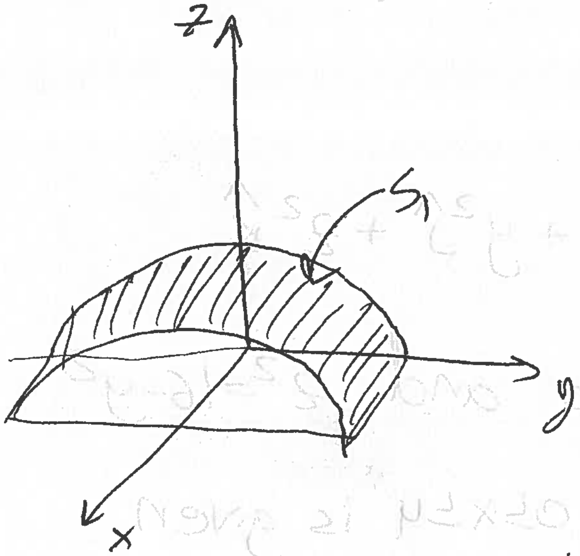
S_3 : front surface (flat semi-circle at $x=$

S_4 : back surface (flat semi-circle at $x=0$)

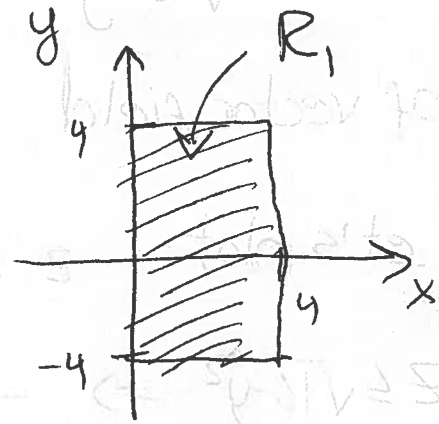
$$\text{Flux} = \underbrace{\sum_{k=1}^{4} \iint_{S_k} \vec{F}\cdot\hat{n}\,ds}_{\text{sum of fluxes through 4 sides}}$$

For surface S_1:

$z = \sqrt{16-y^2}$; $0 \le x \le 4$, $-4 \le y \le 4$ $\Rightarrow y^2+z^2=16$



project along $\hat{p}=\hat{k}$ $\Rightarrow$ onto xy plane



The surface function is ~~is~~ given by:

$g(x,y,z) = y^2+z^2 = 16 \Rightarrow \nabla g = \langle 0, 2y, 2z \rangle$

$\hat{p} = \hat{k} \Rightarrow |\nabla g \cdot \hat{p}| = |\langle 0,2y,2z\rangle \cdot \langle 0,0,1\rangle| = |2z|$

But $z \ge 0 \Rightarrow |\nabla g \cdot \hat{p}| = 2z$

$$\text{Flux}_{S_1} = \iint_{S_1} \vec{F}\cdot\hat{n}\, d\sigma = \iint_{R_1} \vec{F}\cdot \underbrace{\frac{\nabla g}{\|\nabla g\|}}_{\substack{\hat{n} \\ \text{unit normal}}} \cdot \underbrace{\frac{\|\nabla g\|}{|\nabla g\cdot\hat{p}|}}_{\substack{\text{Jacobian element} \\ \text{for change of vars.}}} dA$$

$$= \iint_{R_1} \vec{F}\cdot \frac{\nabla g}{|\nabla g \cdot \hat{p}|}\, dA$$

Note: ∇g is $\perp$ to level curves $g(x,y,z)=C$ by properties of gradient. Hence,

$$\hat{n} = \frac{\pm \nabla g}{\|\nabla g\|}$$ (divide by $\|\nabla g\|$ to make it unit vector)

$\nabla g = \langle 0, 2y, 2z \rangle \Rightarrow \nabla g(1,1,1) = \langle 0,2,2 \rangle$

This points up as it should so we keep $\hat{n} = \frac{+\nabla g}{\|\nabla g\|}$

$$\Rightarrow \text{Flux}_{S_1} = \iint_{R_1} \vec{F} \cdot \frac{\nabla g}{|\nabla g \cdot \hat{p}|}\, dA = \int_{-4}^{4}\int_{0}^{4} \langle x^2, y^2, z^2 \rangle \cdot \frac{\langle 0, 2y, 2z \rangle}{2z}\, dx\, dy$$

$$= \int_{-4}^{4}\int_{0}^{4} \left(\frac{y^3 + z^3}{z} \right) dx\, dy$$

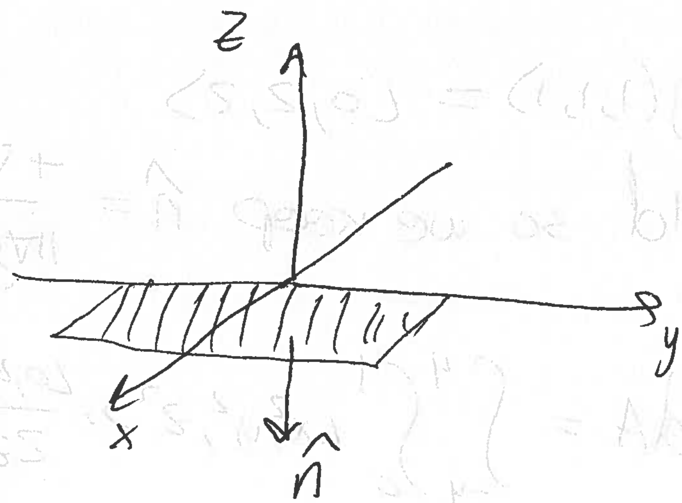
Now replace z by function of x and y since R_1 is in the xy-plane.

$$\Rightarrow \text{Flux}_{S_1} = \int_{-4}^{4}\int_{0}^{4} \left(\frac{y^3}{\underbrace{\sqrt{16-y^2}}_{z}} + \underbrace{(16-y^2)}_{z^2} \right) dx\, dy$$

$$= 4\int_{-4}^{4} \left[\frac{y^3}{\sqrt{16-y^2}} + (16-y^2) \right] dy = \frac{1024}{3}$$

S_2: bottom surface lies in plane $z=0$
(xy-plane)

Let's do quick calculation. We know at bottom normal is $\hat{n} = \langle 0,0,-1 \rangle$

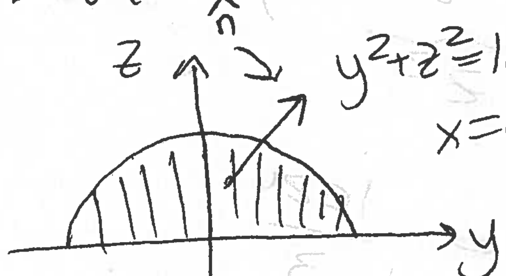


$\vec{F}\cdot\hat{n} = \langle x^2, y^2, z^2 \rangle \cdot \langle 0,0,-1 \rangle = -z^2$

But $z=0 \Rightarrow \vec{F}\cdot\hat{n} = 0$

$\Rightarrow$ so flux will be zero through this surface
no work required!

S_4: back semicircle surface at $x=0$:



$y^2+z^2=16$, $z \geq 0$
$x=0$

we know again that $\hat{n} = \langle -1,0,0 \rangle$

This means

$$\vec{F}\cdot\hat{n} = \langle x^2, y^2, z^2\rangle \cdot \langle -1,0,0\rangle = -x^2$$

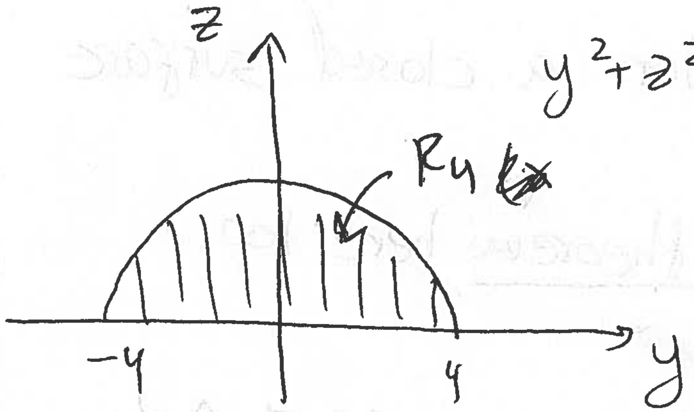
But $x=0$ here so $\vec{F}\cdot\hat{n}=0$
(on S_4)

$\Rightarrow$ flux through S_4 is zero.

Notice: S_4 lies in plane $x=0 \Rightarrow g(x,y,z)=x=0$

$$\hat{n} = \frac{\pm\nabla g}{\|\nabla g\|} = \frac{\pm\langle 1,0,0\rangle}{1}$$ do not need to guess

S_3: this one will have nonzero flux since it's at $x=4$.



$y^2+z^2=16$, $x=4$

we project onto yz-plan
Note that $g(x,y,z)=x=4$
and $\hat{p}=\hat{c}$

$\Rightarrow |\nabla g\cdot\hat{p}| = |\langle 1,0,0\rangle\cdot\langle 1,0,0\rangle| =$
and $\|\nabla g\| = 1$.

$$\text{flux}_{S_4} = \iint_{S_4} \vec{F}\cdot\hat{n}\, d\sigma$$

$$= \iint_{R_4} \vec{F}\cdot\hat{n}\, dA = \iint_{R_4 \text{ @ } x=4} \langle x^2,y^2,z^2\rangle\cdot\langle 1,0,0\rangle\, dA$$

replace x by numeric or y and z function $= \iint_{R_4} x^2\, dA = 16\iint_{R_4} dA = 16\,(\text{area of } R_4)$

Note that

$$\text{Area}(R_4) = \int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}} dz\,dy = \underbrace{\frac{\pi(4)^2}{2}}_{\text{just area of } \frac{1}{2} \text{ circle of radius } 4.}$$

$$\Rightarrow \text{flux}_{S_4} = 16 \cdot \frac{\pi \cdot 16}{2} = 128\pi$$

$$\Rightarrow \text{total flux} = \frac{1024}{3} + 0 + 128\pi + 0 = \frac{1024}{3} + 128\pi$$

Note: positive flux leaves a closed surface
negative flux enters a closed surface

Note: Can use divergence theorem here too.
Think about it.

$$\text{div}\vec{F} = \nabla\cdot\vec{F} = 2x + 2y + 2z \qquad \text{flux} = \iint_S \vec{F}\cdot\hat{n}\,ds =$$

$$= \iiint_E (\nabla\cdot\vec{F})\,dV = \iiint_E (2x+2y+2z)\,dV$$

$$= 2\int_0^4\int_{-4}^{4}\int_0^{\sqrt{16-y^2}} (x+y+z)\,dz\,dy\,dx = \frac{1024}{3} + 128\pi$$