Covered Material and Resources for Part2 (Exam 1 - Exam 2)

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1 September 17

Covered Section 11.1 - functions of two variables, domain, range, good example: find domain and range of $z = f(x, y) = \sqrt{4x^2 - y}$.

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf
- Paul's notes: http://tutorial.math.lamar.edu/Classes/CalcIII/MultiVrbleFcns.aspx

2 September 19

Covered Section 11.2 - limits, definition with epsilon-delta, proving a limit does not exist (this is generally easier than finding a limit which exists - approach the point (a, b) along different directions and show the limit along these directions takes on different values), finding a limit which does exist (see Corral's bounded function way), continuity: for this we must have that $f(a, b)$ is defined and that $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. First check that function is defined at point then see if limit looks like it doesn't exist.

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf
- Paul's notes: http://tutorial.math.lamar.edu/Classes/CalcIII/Limits.aspx

3 September 22

Covered Section 11.3 - partial derivatives, equality of mixed partials

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf

- Paul's notes: http://tutorial.math.lamar.edu/Classes/CalcIII/Limits.aspx
- Khan academy: http://www.khanacademy.org/math/multivariable-calculus/partial_derivatives_topic/partial_derivatives/v/partial-derivatives

4 September 26

Reviewed limits and continuity. Example with using polar coordinates and L'Hospitals rule - for instance, to evaluate:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

By substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$, $r^2 = x^2 + y^2$ and the limit simplifies to $\lim_{r\to 0} \frac{\sin(r^2)}{r^2} = \lim_{r\to 0} \frac{2r\cos(r^2)}{2r} = cos(0) = 1$. Implicit differentiation, reviewed mixed partials.

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf
- Paul's notes: http://tutorial.math.lamar.edu/Classes/CalcIII/Limits.aspx
- youtube: https://www.youtube.com/watch?v=rSRxxTJ9KD8 and https://www.youtube.com/watch?v=YmsZRQJJMG8

5 September 29

Section 11.4 - tangent plane (determined by tangent lines whose slopes are given by partial derivatives), linear approximation using tangent plane, differentials

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf
- Paul's notes: http://tutorial.math.lamar.edu/Classes/CalcIII/TangentPlanes.aspx
- youtube: https://www.youtube.com/watch?v=On1il3WvtXU

6 October 1st

Section 11.5 - chain rule, also reviewed differentials (approximating change in value of $f(x, y)$ when x and y are perturbed - that is $dz \approx f(x + \Delta x, y + \Delta y) - f(x, y)$ where the approximation is good when Δx and Δy are small. Note that dz represents the total change, not the relative change. When talking about relative change or percent error we consider the term $\frac{dz}{z}$ as approximation to $\frac{\Delta z}{z}$. We did an example with the area of the rectangle $A(x, y) = xy \implies A(x + \Delta x, y + \Delta y) = A + \Delta A =$

$(x+\Delta x)(y+\Delta y)$ where the difference between the differential and the true difference in area is the factor $(\Delta x)(\Delta y)$. Differentiation of various composite functions $f(x(t,s), y(t,s))$ via chain rule.

other references:

- Paul's notes: http://tutorial.math.lamar.edu/Classes/CalcIII/ChainRule.aspx
- some solved problems: http://math.sci.ccny.cuny.edu/document/show/2213

7 October 3rd

Section 11.6 - directional derivatives and gradient vector, also reviewed some chain rule examples.

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf
- Paul's notes: http://tutorial.math.lamar.edu/Classes/CalcIII/GradientVectorTangentPlane.aspx
- youtube: https://www.khanacademy.org/math/multivariable-calculus/partial_derivatives_topic/gradient/v/gradient-1, https://www.youtube.com/watch?v=XZ1QwS1IKgw

8 October 6th

Review of gradients from 11.6. Example with sphere $x^2+y^2+z^2=1$. Notice that we can write $F(x,y,z)=x^2+y^2+z^2$. Then the sphere is a level surface of F, i.e. all points (x,y,z) which satisfy $F(x,y,z)=1$. The gradient of F which is a vector $\langle 2x, 2y, 2z\rangle$ is then at any point on the sphere, perpendicular to all curves going through that point (perpendicular to their tangent vector at that point). Notice, however, that we can write the top half sphere as a function $z=f(x,y)=\sqrt{1-x^2-y^2}$. We can then compute the gradient of this function: $\nabla f(x,y) = -\frac{1}{\sqrt{1-x^2-y^2}}\langle x, y\rangle$. This means that at any point (a,b) inside the domain $x^2+y^2\leq 1$, the gradient points in the direction of the origin $(0,0)$ (the center of the circle) which corresponds to the highest point on the sphere (greatest z): $(0,0,1)$. We must head (in the $x-y$ plane) radially inward towards origin to have $z=f(x,y)$ increase the fastest. Notice that the negative of the gradient is the direction radially away from the center of the circle, towards the value $z=0$ when $x^2+y^2=1$ on the outer rim of the half sphere, so the negative gradient of $f(x,y)$ points in the direction of the fastest way to decrease $z=f(x,y)$.

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf

9 October 8th

Section 11.7 - classification of maxima and minima of functions $z = f(x, y)$. Before getting to max/min we reviewed gradients and talked about parabolic reflectors. These are used for years in radio communications, radar, and car or bike lights. For min/max, here is the procedure for identifying local min/max of $z = f(x, y)$. Note that local is not same as global. Even if there is a single critical point of f on a given domain, and this point corresponds to a local min, it does not imply anything about it being a global min. For the procedure, first find critical points which satisfy $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = 0$. Let us say we found such point (a, b) (i.e. $f_x(a, b) = 0$ and $f_y(a, b) = 0$). Obviously there maybe more than one critical point or no critical point. Then, compute the determinant of the Hessian matrix at the critical point.

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$

where we have used the equality of mixed partials (when they are continous). Notice that:

$$D > 0 \implies D = f_{xx}f_{yy} - (f_{xy})^2 > 0 \implies f_{xx}f_{yy} = D + (f_{xy})^2 > 0 \implies f_{xx}f_{yy} > 0$$

The last inequality implies that when $D > 0$, f_{xx} and f_{yy} are of the same sign (either both positive or both negative. Hence, it is sufficient below to test if one of them is less than or greater than 0.)

After the Hessian is computed follow the algorithm - you can only conclude (a, b) is local min or max when $D > 0$. When $D < 0$, you conclude that (a, b) is neither local min nor local max and when $D = 0$ you can't conclude anything from this test (it could be local min or max or could not be).

- (A) If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is local min.
- (B) If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is local max.
- (C) If $D < 0$ then f has neither local min nor local max at (a, b).
- (D) If $D = 0$ this test gives no information.

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf
- Pauls notes: http://tutorial.math.lamar.edu/Classes/CalcIII/RelativeExtrema.aspx
- Note: http://www.math.tamu.edu/~tvogel/gallery/node16.html
- Youtube: https://www.youtube.com/watch?v=UQrKtiKfmMU

10 October 10th

Finished Section 11.7 - local min/max and absolute min/max on closed domain. Reviewed algorithm for finding local/min max of functions above. Note that the local min/max found this way are usually strictly local (i.e. if (a, b) is a local min, $f(a, b) \leq f(x, y)$ for points (x, y) around the point (a, b) only, not for all (x, y)). When f is defined on a closed domain (such as a disk with boundary for example), then there exist global maxima and minima on this domain: either at the critical points of f inside the domain, or on the boundary. We did an example in class with a disk. Thus, there are three classes of problems for finding min/max: (1) given $f(x, y)$ find local min and max by finding critical points and using the determinant and second derivative based test, (2) given $f(x, y)$ defined on some closed domain, find global min and max by finding critical points on domain and comparing value of f at these critical points to max/min values of f on the boundary, and (3) given $f(x, y)$ and a constraint condition $g(x, y) = C$, find minimum and maximum of f using the method of Lagrange multipliers.

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf
- Pauls notes: http://tutorial.math.lamar.edu/Classes/CalcIII/RelativeExtrema.aspx and http://tutorial.math.lamar.edu/Classes/CalcIII/AbsoluteExtrema.aspx
- Youtube: https://www.youtube.com/watch?v=vkzsxvucLjQ, https://www.youtube.com/watch?v=Hm5QnuDjNmY

11 October 13th

Section 11.8 - Lagrange multipliers. Used for solving optimization problems with a constraint, for instance: prove that the rectangle with max area that has perimeter p is a square. Given a rectangle of sides x and y, we set up the function we want to extremize: $A = f(x, y) = xy$ and the constraint $g(x, y) = 2x + 2y = p$. Then we solve $\nabla f = \lambda \nabla g$ with λ a constant, to identify constrained points:

$$\langle y, x \rangle = \lambda \langle 2, 2 \rangle \implies \lambda = \frac{y}{2} = \frac{x}{2} \implies x = y$$

Plugging into constraint, we have:

$$2x + 2y = 2x + 2x = 4x = p \implies x = \frac{p}{4}$$

We conclude that the largest area is that of a $\frac{p}{4} \times \frac{p}{4}$ square, with area $f(\frac{p}{4}, \frac{p}{4}) = \frac{p^2}{16}$. Note that the Lagrange multiplier approach is not guaranteed to give a global minimum or maximum solution. If you have a closed boundary, then you must also evaluate the function you want to extremize (maximize or minimize) at the boundary points, in addition to constrained points you find with the Lagrange multiplier approach. Then you pick max/min from amongst all these. See hw 7, rod problem. Also note that in some problems there are multiple constraints: i.e. $g_1(x, y) = C_1, g_2(x, y) = C_2$, etc. Then you have $\nabla f(x, y) = \lambda \nabla g_1(x, y) + \mu \nabla g_2(x, y)$. See textbook an Paul's notes for examples.

other references:

- Vector Calculus by Corral: http://www.mecmath.net/calc3book.pdf
- Paul's notes: http://tutorial.math.lamar.edu/Classes/CalcIII/LagrangeMultipliers.aspx

12 October 15th

More examples with Lagrange multipliers and power series. Power series for functions of two variables is simply the expansion of $f(x, y)$ around some point (a, b). Here are the linear and quadratic terms:

$$\begin{aligned} f(x,y) &\approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \\ &\frac{1}{2!}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right) \end{aligned}$$

Note that the first linear part $f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$ is just the Tangent line linearized approximation we saw before (i.e. $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is the equation of the tangent plane to $z = f(x, y)$ at (a, b)). Including the quadratic term (and other higher order terms) gives a more accurate approximation to $f(x, y)$.

other references:

- Youtube: https://www.youtube.com/watch?v=Z7yTsoMTjnM, https://www.youtube.com/watch?v=SJx11bTUqw8

13 October 17th

More on Power/Taylor Series and Lagrange multiplier example with two constaints. Expanding $f(x, y)$ around point (a, b) using two terms (linear and quadratic) and the cubic remainder term yields:

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \\ &\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \\ &\frac{1}{3!}\left[(x-a)^3 f_{xxx}(c,d) + 3(x-a)^2(y-b)f_{xxy}(c,d) + 3(x-a)(y-b)^2 f_{xyy}(c,d) + (y-b)^3 f_{yyy}(c,d)\right] \end{aligned}$$

where in the last term (c, d) is some point between (a, b) and (x, y). This is the remainder from Taylor's theorem in two dimensions. The remainder is just the error you get from approximating $f(x, y)$ with a finite number of Taylor series terms around (a, b). Notice that the above formula allows us to put an upper bound on the error or remainder term (from taking only the first two terms of the Taylors series), if we know the quantities $|x-a|, |y-b|, ||f_{xxx}||_\infty, ||f_{xxy}||_\infty, ||f_{xyy}||_\infty, ||f_{yyy}||_\infty$ where the max norms are the max absolute values of the partial derivatives in the interval between (x, y) (where you want to approximate f) and (a, b) (the point around which you write the Taylor expansion). For simple functions $f(x, y)$ is it typically not difficult to bound these quantities.

other references:

- Gatech notes: http://people.math.gatech.edu/~cain/notes/calculus.html