

Covered Material and Resources for Part1 (0 - Exam 1)

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1 I: Intro to ODE/IVPs, phase plots

Textbook: Section 1.1 (intro to ode, modeling), 1.2 (phase plots)

Introduction to ODE (ordinary differential equation) and IVP (initial value problem) which consists of ODE and an initial condition. Order of equation is the highest order derivative that appears. Without initial conditions, there are usually infinitely many solutions (solution involves arbitrary constant C). When an initial condition is present, we can have the case of a unique solution, many solutions, or no solutions at all. Plotting phase plots for first order equations. I.e. in the general first order equation $y'(t) = f(t, y)$, even if we cannot solve for $y(t)$, we still know the slope of the solution at each time t , it is given by the function $f(t, y)$. Note that t is just the independent variable, x is also frequently used. We can make a plot of slopes in the $t - y$ plane. We can then plot solution curves in this plane (there are many solution curves stemming from different initial conditions) but all solution curves must be tangent to the slope field lines in the region they pass through. This allows us to predict long term behavior of solutions as $t \rightarrow \infty$ and also characterize equilibrium solutions (if there are any). Equilibrium solutions are constant solutions independent of t , $y(t) = C$ to the ODE (which when plotted in the $t - y$ plane is just a line). Obviously this implies that $y'(t) = 0$. Note that equilibrium solutions can be stable, semistable or unstable. The equilibrium solution is stable if on both sides of $y = C$ (above and below) the slope fields point towards the line. If the slope fields point away from the line on different sides, then the equilibrium solution is unstable. If one side points towards the line and the other side away, the equilibrium is called semi-stable. If slope fields point toward the equilibrium line on all side then the equilibrium solution is stable. Slope fields aid in the analysis of first order ode: constant solutions, straight line solutions, concavity of solutions (where $y'' > 0$ and where $y'' < 0$). What happens when $t \rightarrow \infty$.

See slope field examples later in this pdf.

Other references:

- Diffeq for Engineers book (1-1.3) <http://www.jirka.org/diffyqs/diffyqs.pdf>
- Paul's Notes <http://tutorial.math.lamar.edu/Classes/DE/DirectionFields.aspx>, <http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx>, <http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx>

math.lamar.edu/Classes/DE/Separable.aspx, <http://tutorial.math.lamar.edu/Classes/CalcII/PartialFractions.aspx>

- William F. Trench book (pg. 1 - 33) <http://digitalcommons.trinity.edu/mono/8/>
- UC Davis Lectures (ch. 1) <http://www.math.ucdavis.edu/~tracy/courses/math22B/22BBook.pdf>

2 II: Separation of Variables, Picard's theorem, Numerical Methods

Textbook: 1.2-1.3 (separation of variables), 1.4 (numerical methods), 1.5 (Picard's theorem)

A differential equation (linear or nonlinear) is separable, if you can move all terms involving the independent variable to one side and all terms involving the dependent variable to the other side. Examples are exponential growth:

$$\frac{dy}{dt} = ky \implies \frac{dy}{y} = kdt \implies \int \frac{1}{y} dy = \int kdt \implies \ln|y| = kt + C_1 \implies y(t) = Ce^{kt}$$

and nonlinear growth:

$$\begin{aligned} y' - \tan(x)y = 0 &\implies \frac{dy}{y} = \tan(x)dx \implies \ln|y| = \ln|\sec(x)| + C_1 \\ &\implies |y(x)| = e^{(\ln|\sec(x)| + C_1)} = e^{\ln|\sec(x)|} e^{C_1} = |C_2| |\sec(x)| \end{aligned}$$

Since C_2 can be positive or negative, we can remove the sign on the absolute value of $|y(x)|$ to get $y(x) = C|\sec(x)|$ for some constant C . Often one needs to use integration by parts or partial fractions to do the integrations. This is the case in the derivation of the logistic equation and similar differential equations. For example:

$$\frac{1}{y(2-y)} = \frac{A}{y} + \frac{B}{2-y} \implies 1 = A(2-y) + By$$

Plugging in $y = 0$, we get $1 = 2A$ from which $A = \frac{1}{2}$. Also, plugging in $y = 2$, we get $1 = 2B$ from which $B = \frac{1}{2}$. So the decomposition is:

$$\frac{1}{y(2-y)} = \frac{1}{2y} + \frac{1}{2(2-y)}$$

Also, recall the various properties of logarithms (for example, $\log(b) - \log(a) = \log\left(\frac{b}{a}\right) \implies \log(a^{-1}) = -\log(a)$) and of exponential (for example, $e^{a+b} = e^a e^b$), they are frequently used in calculations.

More examples on separation of variables with variable substitution, as in $\frac{dy}{dt} = f\left(\frac{y}{t}\right)$. To turn this into a separable equation, set $v = \frac{y}{t}$ to turn this into a separable equation for v :

$$y = tv \implies \frac{dy}{dt} = v + t \frac{dv}{dt} = f(v) \implies \frac{dv}{f(v) - v} = \frac{dt}{t}$$

Picard's theorem for testing existence and uniqueness of first order IVP: $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$. Check if $f(t, y)$ and $\frac{\partial}{\partial y}f(t, y)$ are continuous in some rectangle around the point (t_0, y_0) . Continuity of $f(t, y)$ implies existence and continuity of $\frac{\partial}{\partial y}f(t, y)$ implies uniqueness, all in some interval of (t_0, y_0) . That is, even if $f(t, y)$ and $\frac{\partial}{\partial y}f(t, y)$ are continuous everywhere, the solution $y(t)$ may only be defined in some interval of t containing t_0 (i.e. on an open interval $(t_0 - \sigma_1 < t_0 < t_0 + \sigma_1)$). Note that Picard's theorem only gives necessary condition for existence (the continuity of $f(t, y)$ in some rectangle around (t_0, y_0)) and for uniqueness (the continuity of both $f(t, y)$ and of $\frac{\partial}{\partial y}f(t, y)$ in some rectangle around (t_0, y_0)). If for example, $f(t, y)$ and $\frac{\partial}{\partial y}f(t, y)$ are continuous everywhere except at $t = 0$ then this means that the IVP with $y(t_0) = y_0$ and $t_0 \neq 0$ will have a unique solution $y(t)$ defined on some interval $(t_0 - \sigma_1 < t_0 < t_0 + \sigma_1)$. It could be defined also on the whole line, we do not know from the theorem. However, at $t = 0$ if $f(t, y)$ and $\frac{\partial}{\partial y}f(t, y)$ are not continuous there, then we have no information from Picard's theorem. We are not guaranteed existence or uniqueness and anything can happen there. The solution may or may not exist and if it exists it may or may not be unique. Picard's theorem (sufficient conditions for existence and uniqueness based on continuity of $f(t, y)$ and $\frac{\partial}{\partial y}f(t, y)$) is one sided: that is, failure to satisfy the continuity conditions does not imply that the solution does not exist or is not unique. However, if for the ODE the function $f(t, y)$ is not defined for the particular (t_0, y_0) given by the initial condition then the IVP is not defined for this IC and there is no solution (this is not an argument of Picard's theorem however).

As an example, consider the IVP:

$$y' = \frac{2y}{x} \quad ; \quad y(x_0) = y_0 \tag{2.1}$$

We can solve the ODE in (2.1) using separation of variables:

$$\frac{dy}{y} = 2 \frac{dx}{x} \implies \ln|y| = 2 \ln|x| + C_1 = \ln x^2 + C_1 \implies y(x) = Cx^2$$

Now for Picard's theorem, we consider the continuity of $f(x, y) = \frac{2y}{x}$ and of $\frac{\partial}{\partial y}f(x, y) = \frac{2}{x}$ around (x_0, y_0) . Clearly, both are continuous everywhere for $x \neq 0$. This implies that Picard's theorem guarantees a unique solution for the IC $y(x_0) = y_0, x_0 \neq 0$. The unique solution is $y(x) = \frac{y_0}{x_0^2}x^2$. On the other hand when $x_0 = 0$, Picard's theorem does not apply and we cannot conclude anything right away. From the solution $y = Cx^2$, notice that when $x_0 = 0$ and $y_0 \neq 0$, there is no such constant C which satisfies $y_0 = C \times 0$, so that implies that there is no solution in this case. In fact, clearly, $f(x, y)$ is undefined for $x = 0$. However, note that when both $x_0 = 0$ and $y_0 = 0$, the solution $y = Cx^2$ holds for any constant C . Hence, we see that for the same differential equation, but variable initial conditions, different outcomes are possible depending on the initial condition.

Numerical Methods: see supplement later in this file for information on Euler's and Runge Kutta Methods.

Other references:

- Online pdfs: www.math.uiuc.edu/~tyson/existence.pdf, www.math.ku.edu/~lerner/m221f12/LinearDEs.pdf

- Paul's Notes <http://tutorial.math.lamar.edu/Classes/DE/EulersMethod.aspx>
- DiffEq for Engineers <http://www.jirka.org/diffyqs/diffyqs.pdf>
- William F. Trench book <http://digitalcommons.trinity.edu/mono/8/>
- UC Davis Lectures <http://www.math.ucdavis.edu/~tracy/courses/math22B/22BBook.pdf>

3 Linear Operators and Linear Differential Equations, Integrating Factors, Variation of Parameters

Textbook: 2.1 (linear operators and equations), 2.2 (first order linear equations), 2.3-2.4 (applications of first order ODEs)

A differential equation can be linear or nonlinear. The advantage of linear equations is that they are typically much easier to solve and their behavior less complex. Many of the analytical techniques we will discuss are designed specifically for linear equations. We can write linear differential equations in operator form: $L(y) = 0$ or $L(y) = f(t)$. The first form corresponds to a homogeneous ODE and the second form to an inhomogeneous one. Notice that a linear equation of n -th order can always be written in the form:

$$a_n(t) \frac{d^n y}{dt^n} + \cdots + a_1(t) \frac{dy}{dt} + a_0(t)y = f(t) \quad (3.1)$$

where the functions $a_n(t), \dots, a_1(t), a_0(t)$ are not necessarily linear functions of t (i.e. we can have $a_1(t) = \sin(t)$). Equation (3.4) is called homogeneous whenever $f(t) = 0$ for all t . If $f(t) \neq 0$, then the corresponding homogeneous equation to $L(y) = f(t)$ above is $L(y) = 0$. The operator corresponding to (3.4) is:

$$L = a_n(t) \frac{d^n}{dt^n} + \cdots + a_1(t) \frac{d}{dt} + a_0(t) \quad (3.2)$$

when we apply this operator to the function $y(t)$ (the ODE solution), we obtain the ODE:

$$L(y) = a_n(t) \frac{d^n y(t)}{dt^n} + \cdots + a_1(t) \frac{dy(t)}{dt} + a_0(t)y(t) = f(t) \quad (3.3)$$

Notice that for $n = 1$ (first order ODEs), the linear ODEs can always be written in the form $y'(t) + p(t)y(t) = f(t)$ for some function $p(t)$ (possibly zero or nonlinear, etc).

An operator L is linear if $L(Cy) = CL(y)$ and $L(u+v) = L(u) + L(v)$. These two properties imply that for constants C_1 and C_2 and functions u and v we have that $L(C_1u + C_2v) = C_1L(u) + C_2L(v)$. An example of a linear operator is $L(y) = 2y + C$. A differential linear operator is an operator involving derivatives. Notice that a linear differential equation of n -th order can always be written in the form:

$$a_n(t) \frac{d^n y}{dt^n} + \cdots + a_1(t) \frac{dy}{dt} + a_0(t)y = f(t) \quad (3.4)$$

where the functions $a_n(t), \dots, a_1(t), a_0(t)$ are not necessarily linear functions of t (i.e. we can have $a_1(t) = \sin(t)$). Equation (3.4) is called homogeneous whenever $f(t) = 0$ for all t . If $f(t) \neq 0$, then the

corresponding homogeneous equation to $L(y) = f(t)$ above is $L(y) = 0$. The operator corresponding to (3.4) is:

$$L = a_n(t) \frac{d^n}{dt^n} + \dots + a_1(t) \frac{d}{dt} + a_0(t) \quad (3.5)$$

when we apply this operator to the function $y(t)$ (the ODE solution), we obtain the ODE:

$$L(y) = a_n(t) \frac{d^n y(t)}{dt^n} + \dots + a_1(t) \frac{dy(t)}{dt} + a_0(t)y(t) = f(t) \quad (3.6)$$

so we can express linear differential equations in the form $L(y) = f(t)$. Again, for linearity in differential equations we are concerned with the independent variable y not the dependent variable t . So notice that $y'' + t^2 y' = \sin(t)$ is a linear ODE (because the differential operator in y is linear, even though t^2 and $\sin(t)$ are nonlinear) while the ODE $y'' + t^2 y' = \sin(y)$ is nonlinear because $\sin(y)$ is nonlinear in y .

Properties of linear operators and equations. Notice that the linear equation $y'' + t^2 y' = \sin(t)$ can be written as $L(y) = f(t)$ with $L = \frac{d^2}{dt^2} + t^2 \frac{d}{dt}$ and $f(t) = \sin(t)$. Superposition of solutions. Notice that if for the homogeneous differential equation $L(u) = 0$, y_1 and y_2 are solutions, that is $L(y_1) = 0$ and $L(y_2) = 0$, then $L(C_1 y_1 + C_2 y_2) = C_1 L(y_1) + C_2 L(y_2) = 0 + 0 = 0$, so any linear combination of the two solutions (i.e. the function $C_1 y_1(t) + C_2 y_2(t)$) is also a solution. Of course, if we have $L(u) = f$ (non-homogeneous equation), this will not hold.

For the nonhomogeneous linear differential equation $L(y) = f(t)$, the general solution is always of the form $y(t) = y_h(t) + y_p(t)$ where $y_h(t)$ solves the corresponding homogeneous equation $L(y) = 0$ and $y_p(t)$ is one particular solution to the nonhomogeneous equation. To see this, let $y_h(t)$ satisfy $L(y_h) = 0$ and $y_p(t)$ some particular solution satisfying $L(y_p) = f(t)$. Then it always holds that $y(t) = y_p(t) + (y(t) - y_p(t))$. By linearity of L , we have that $L(y) = L(y_p) + L(y) - L(y_p) = f(t) + f(t) - f(t) = f(t)$. Thus, $y(t)$ satisfies the non-homogeneous differential equation. It is the sum of the particular solution $y_p(t)$ and of $y(t) - y_p(t)$ which satisfies the homogeneous equation since $L(y - y_p) = L(y) - L(y_p) = f(t) - f(t) = 0$.

There are two main methods for finding solutions to non-homogeneous first order linear ODEs $y'(t) + p(t)y(t) = f(t)$: variation of parameters and integrating factors.

Let's review variation of parameters. If the ODE is first order and linear it can be put in the form

$$y'(t) + p(t)y(t) = f(t) \quad (3.7)$$

When $f(t) \neq 0$, the equation is non-homogeneous. The general solution is of the form $y(t) = y_h(t) + y_p(t)$ where $y_h(t)$ satisfies $y'_h(t) + p(t)y_h(t) = 0$ and $y_p(t)$ satisfies $y'_p(t) + p(t)y_p(t) = f(t)$ (one particular solution). The homogeneous solution is easy to find:

$$y'_h(t) + p(t)y_h(t) = 0 \implies \frac{dy}{dt} = -py \implies y_h(t) = C e^{\int (-p(t)) dt}$$

Then for the particular solution set $y_p(t) = v(t)e^{\int (-p(t)) dt}$, replacing the constant C in the homogeneous solution by a function of t and plug into the ODE. Upon substitution into $y'_p(t) + p(t)y_p(t) = f(t)$, we obtain:

$$v' e^{\int (-p(t)) dt} = f(t) \implies v(t) = \int f(t) e^{\int p(t) dt} \implies y_p(t) = v(t) e^{\int (-p(t)) dt} = \left[\int f(t) e^{\int p(t) dt} \right] e^{\int (-p(t)) dt}$$

where we used that $y'_p(t) = v'e^{\int(-p(t))dt} + v(t)e^{\int(-p(t))dt}[-p(t)]$ to obtain cancellations with the term $p(t)y_p(t)$ of the ODE. So the general solution to (3.7) is given by the sum of the solution to the homogeneous equation (with a constant C) and the derived particular solution:

$$y(t) = y_h(t) + y_p(t) = Ce^{\int(-p(t))dt} + \left[\int f(t)e^{\int(p(t))dt} \right] e^{\int(-p(t))dt} \quad (3.8)$$

Let's review the method of integrating factors. Again, we assume your ODE is first order and linear (otherwise this technique does not work in this form) and can be put in the form $y'(t) + p(t)y(t) = f(t)$. Let us find a function $\mu(t)$ such that when we multiply both sides of the ODE (in the standard form (3.7)) by $\mu(t)$, we get:

$$\mu(t) (y'(t) + p(t)y(t)) = \mu(t)f(t) \text{ and } \mu(t) (y'(t) + p(t)y(t)) = \frac{d}{dt} (\mu(t)y(t))$$

To make sure $\mu(t)$ satisfies the second equality, we expand to get:

$$\frac{d}{dt} (\mu(t)y(t)) = \mu'(t)y(t) + \mu(t)y'(t) = \mu(t)y'(t) + p(t)\mu(t)y(t) \implies \mu'(t)y(t) = p(t)\mu(t)y(t)$$

We assume $y(t) \neq 0$ so that $\mu(t)$ satisfies:

$$\frac{\mu'(t)}{\mu(t)} = p(t) \implies \ln |\mu(t)| = \int p(t)dt \implies \mu(t) = e^{\int p(t)dt}$$

We set the integration constant to zero and can assume $\mu(t) > 0$ when we get $\mu(t)$ involving an absolute value, to get as simple as possible $\mu(t)$. Thus, with this choice of $\mu(t)$ we can solve for the general solution $y(t)$:

$$\frac{d}{dt} (\mu(t)y(t)) = \mu(t)f(t) \implies \mu(t)y(t) = \int (\mu(t)f(t)) dt + C \implies y(t) = \frac{1}{\mu(t)} \int (\mu(t)f(t)) dt + \frac{C}{\mu(t)}$$

The constant C here from the integration of $\mu(t)f(t)$ must be included as that will give the homogeneous part of the solution when divided by $\mu(t)$. Notice that when we plug in for $\mu(t)$ we derive exactly the same general solution as given by (3.8), which we derived using variation of parameters method.

As an example let's consider the ODE

$$y' + \frac{4}{x}y = x^4 \quad (3.9)$$

The corresponding homogeneous equation is $y'_h + \frac{4}{x}y_h = 0$ and we get by separating variables:

$$\frac{dy}{dx} = -\frac{4y}{x} \implies \frac{dy}{y} = -4\frac{dx}{x} \implies \ln |y| = -4 \ln |x| + C_1 = \ln |x|^{-4} + C_1 \implies y_h(x) = C|x|^{-4} = Cx^{-4}$$

Note that $|x|$ to even powers is same as x . Having found the solution to the homogeneous equation, we assume by variation of parameters technique that $y_p(x) = v(x)|x|^{-4}$. We plug this into $y'_p + \frac{4}{x}y_p = x^4$ to

get $v'(x)x^{-4} = x^4$. This means that $v'(x) = x^8$ and the simplest $v(x) = \frac{x^9}{9}$ with integration constant zero. So a particular solution is $y_p(x) = x^{-4}\frac{x^9}{9} = \frac{x^5}{9}$. This means the general solution to the ODE is $y(x) = y_h(x) + y_p(x) = Cx^{-4} + \frac{x^5}{9}$.

We can solve the same (3.9) using the method of integrating factors. Note that the integrating factor is $\mu(x) = e^{\int \frac{4}{x} dx} = e^{4 \ln|x|} = e^{\ln|x|^4} = x^4$. This implies that:

$$\frac{d}{dx}(\mu(x)y(x)) = \mu(x)f(x) = x^4x^4 = x^8 \implies x^4y(x) = \frac{x^9}{9} + C \implies y(x) = \frac{x^5}{9} + Cx^{-4}$$

As another example, consider $y' + y = \sin(t)$ which is already in the form $y'(t) + p(t)y(t) = f(t)$ with $p(t) = 1$ and $f(t) = \sin(t)$. We solve this by method of integrating factors:

$$\mu(t)[y' + y] = \frac{d}{dt}(\mu(t)y(t)) = \mu(t)\sin(t)$$

with $\mu(t) = e^{\int p(t)dt} = e^t$. Thus, we get:

$$\frac{d}{dt}(e^t y(t)) = e^t \sin(t) \implies ye^t = \int e^t \sin(t) dt + C = \frac{1}{2}e^t(\sin(t) - \cos(t)) + C$$

where the integral needs to be evaluated using integration by parts (IBP). Thus, after dividing both sides by e^t , we get the general solution $y(t) = Ce^{-t} + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)$.

Applications of first order ODEs: compound interest, Newton's law of cooling, Mixing problems (see supplement later in this pdf).

Other references:

- Online pdfs: www.math.ku.edu/~lerner/m221f12/LinearDEs.pdf, www.math.poly.edu/courses/ma1112/ODE1.pdf
- Paul's Notes: <http://tutorial.math.lamar.edu/Classes/DE/Linear.aspx>
- Wikibook http://en.wikibooks.org/wiki/Ordinary_Differential_Equations/First_Order_Linear_1
- Diffeq for Engineers <http://www.jirka.org/diffyqs/diffyqs.pdf>
- William F. Trench book <http://digitalcommons.trinity.edu/mono/8/>

4 Nonlinear Equations

Textbook: section 2.5

Generally, nonlinear equations are harder to solve than linear ones. For this reason, if we can convert them via variable substitution to a linear equation that is great news. A good example is the so called Bernoulli equation:

$$y' + p(x)y = q(x)y^n \tag{4.1}$$

This equation is nonlinear for $n \neq 0$ and $n \neq 1$ due to the y^n term. However, consider the variable substitution $v = y^{1-n}$. We calculate: $v' = (1-n)y^{1-n-1}y' = (1-n)y^{-n}y'$. If we divide both sides of equation (4.1) by y^n (note we assume $y(x) \neq 0$ since the zero solution trivially satisfies the ode for non-negative n and we like to look for nontrivial solutions), we get:

$$y^{-n}y' + y^{1-n}p(x) = q(x) \quad (4.2)$$

and we can replace all terms involving y in terms of the variable v to get the linear equation:

$$\frac{1}{1-n}v' + p(x)v = q(x) \implies v' + (1-n)p(x)v = (1-n)q(x) \quad (4.3)$$

This equation is linear in v and can be solved by integrating factors or variation of parameters technique. Note that we often want to find equilibrium solutions and inflection point lines of nonlinear equations. Consider for example the logistic ODE:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{L}\right)y, \quad y(0) = y_0, \quad r, L \in \mathbb{R} \quad (4.4)$$

The solution to (4.4) can be derived by using partial fractions and properties of logarithms. Note that $y = 0$ and $y = L$ make the right hand side zero and these are thus equilibrium solutions. On the other hand we can compute the second derivative of y :

$$\frac{d^2y}{dt^2} = \frac{d}{dt} \left(\frac{dy}{dt} \right) = \frac{d}{dt} \left(r\left(1 - \frac{y}{L}\right)y \right) = (ry)\left(-\frac{1}{L}\right)\frac{dy}{dt} + r\frac{dy}{dt}\left(1 - \frac{y}{L}\right) = \frac{dy}{dt} \left(-\frac{2ry}{L} + r \right) = r\left(1 - \frac{y}{L}\right)y \left(-\frac{2ry}{L} + r \right)$$

Note that we have $\frac{d^2y}{dt^2} = 0$ for $y = 0, y = L, -\frac{2ry}{L} + r = 0 \implies y = \frac{L}{2}$. Since $y = 0$ and $y = L$ are equilibrium solutions only $y = \frac{L}{2}$ is a line of inflection points. Notice that by analysis of the signs of the function $f(t, y) = r\left(1 - \frac{y}{L}\right)y$ (assuming $r, L > 0$) we can conclude that $y = L$ is a stable equilibrium solution while $y = 0$ is unstable equilibrium.

Other references:

- Paul's Notes: <http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx>
- Web link: <http://math.usu.edu/~powell/biomath/mlab3-02/node3.html>

Slope Field Examples

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1 Introduction

The solutions to many first order ODEs $y'(t) = f(t, y)$ can be analyzed by means of slope field techniques, even if the analytical solution $y(t)$ is not obtained. This is because even if $y(t)$ is not known, the slope of the solution function $y(t)$ at each point of the $y - t$ plane is given by $f(t, y)$. We can make use of this information to make a plot of the slopes of the solution curves (by solution curves we mean solutions $y(t)$ satisfying the ODE with different initial conditions). Notice that if $f(t, y)$ is a continuous function, then the solution function $y(t)$ will also be continuous, since it is obtained by integrating $f(t, y)$. For this reason, to draw solution curves, we simply must connect the slopes that we draw. The particularly useful use of slope fields is to determine what happens to the solutions as $t \rightarrow \pm\infty$. Notice, that we often need to check the sign of $y''(t) = \frac{d}{dt}f(t, y(t))$ to draw solution curves of the correct concavity.

2 Examples

Ex1: $y' = t + y$

Here, there are no constant solutions $y(t) = C$ (not possible for $t + C = 0$ with constant C), but there are straight line solutions. Plug in $y(t) = C_1 t + C_2 \implies y'(t) = C_1$ into the ODE: $C_1 = t + C_1 t + C_2 \implies C_1 = -1, C_2 = -1$. So $y(t) = -t - 1$ is a straight line solution which satisfies the ODE. To analyze the concavity of the solutions in different regions of the $t - y$ plane, consider the second derivative $y''(t)$:

$$y''(t) = 1 + y' = 1 + t + y = y - (-t - 1)$$

Note that for $y > (-t - 1)$, $y'' > 0$ and solutions are concave up. For $y < (-t - 1)$, $y'' < 0$ and solutions are concave down. That is, all solutions above the straight line solution $y = -t - 1$ are concave up and all those below are concave down. The rest of the questions can be answered by looking at the slope field. Note that to plot the slope field we can use software: for example, use `plotdf(x+y)` in maxima (x takes role of t) or online slope field plotters. However, it can obviously also be sketched by hand. We consider that along the straight line solution $y(t) = -t - 1$, the slopes are always minus one, so we draw this in first. Then note from the ode, that $f(t, y) = t + y = y - (-t)$, so that for $y > -t$, slopes are positive and for $y < -t$ slopes are negative and for $y = -t$ the slopes are zero. Here are

two pictures of the slope field below, along with some solution curves. In the first picture (left), in the shaded region, the solution curves below the line $y = -t - 1$ are concave down. The solution curves are drawn by approximately drawing in curves which are everywhere tangent to the slope field lines (obviously this is not exact, only a rough sketch of what they look like - again, each line corresponds to a different initial condition, as long as $f(t, y)$ and $f_y(t, y)$ are continuous, they won't cross - more on this when we discuss Picard's theorem).

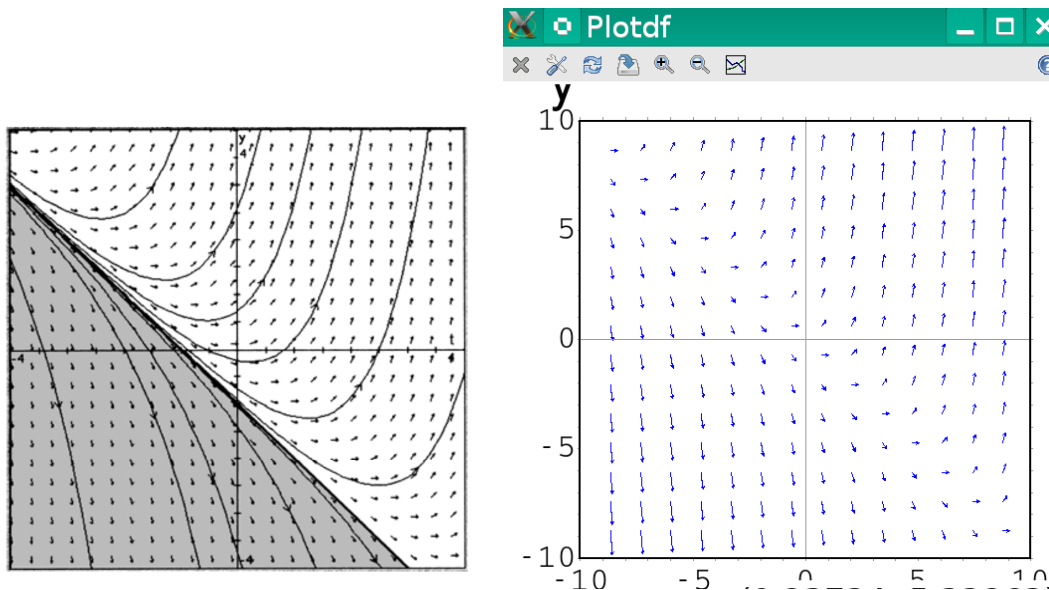


Figure 1: Two versions of the slope field for ODE 1

Now based on the slope fields we analyze the behavior of the solutions to the ODE as $t \rightarrow \infty$ and $t \rightarrow -\infty$. Clearly, as $t \rightarrow \infty$, solutions above the straight line solution $y = -t - 1$ approach ∞ since they are concave up. Solutions below $y = -t - 1$ approach $-\infty$ in this case. As $t \rightarrow -\infty$, all solutions come from (seem to emanate from) ∞ . That is, for $t \rightarrow -\infty$, $y \rightarrow \infty$. This is evident from the slope fields and the drawn in solution curves. Note that for solutions starting on the line $y = -t - 1$ (with IC on the line), they simply stay on the line and travel from $-\infty$ to $+\infty$ as t changes in the same range. The asymptote in this case is the straight line $y = -t - 1$ (we can see from the slope field plot that if we go backward in time, as $t \rightarrow -\infty$, all the solution curves are close to the line $y = -t - 1$).

Ex 2: $y' = \frac{1}{t-y}$. The ODE is undefined along the line $y = t$. Note that there are no constant solutions, but there is a straight line solution. Plug in $y(t) = C_1t + C_2$ into the ODE to get: $C_1 = \frac{1}{t-C_1t-C_2} \implies C_1 = 1, C_2 = -1$ so $C_1t + C_2 = t - 1$ satisfies the ODE. Now note that:

$$y'(t) = (t-y)^{-1} \implies y''(t) = -(t-y)^{-2}(1-y') = -(t-y)^{-2}\left(1 - \frac{1}{t-y}\right) = -\frac{1}{(t-y)^2} \left(\frac{t-y-1}{t-y}\right) = \frac{y-(t-1)}{(t-y)^3}$$

Let's analyze what these tell us about the signs of y' and y'' in different regions of the $t - y$ plane. First, along $y = t$ (and close to this line), the slope approaches $\pm\infty$ (depending on the side of $y = t$ we

are on). So even though the ODE is not defined on the line, immediately around the line the slopes point almost vertically up or down (they point up for $t > y$, immediately below the line $y = t$). Next, for $t > y$, $t - y > 0$, so $y' > 0$ (so slopes are positive below the line $y = t$). On the other hand, for $t < y$ (above the line $y = t$ in the $t - y$ plane), the slopes are negative. Along the straight line solution, the slopes are one. Let us now analyze the concavity of the solutions in different regions of the $t - y$ plane. For this we must look at the signs of: $y''(t) = \frac{y-(t-1)}{(t-y)^3}$ in different regions of the $t - y$ plane. Notice that the sign depends on the sign of the numerator $y - (t - 1)$ and on the sign of the denominator $(t - y)^3$ (which has the same sign as $t - y$). Notice that for $y > (t - 1)$ and $y < t$, we have $y'' = \frac{(+)}{(+)} > 0$, so the solutions are concave up in this region. On the other hand, when $y > (t - 1)$ and $y > t$ or when $y < (t - 1)$ and $y < t$, we have that $y'' < 0$, so the solutions are concave down in this region. The only other case when $y < (t - 1), y > t$ is impossible since $y > t \implies y > (t - 1)$. So we have that the solution curves are concave up only in the region $(t - 1) < y < t$. Essentially all these concern the regions above, below, and between the lines $y = t$ and $y = (t - 1)$. Two pictures of the slope field appear below:

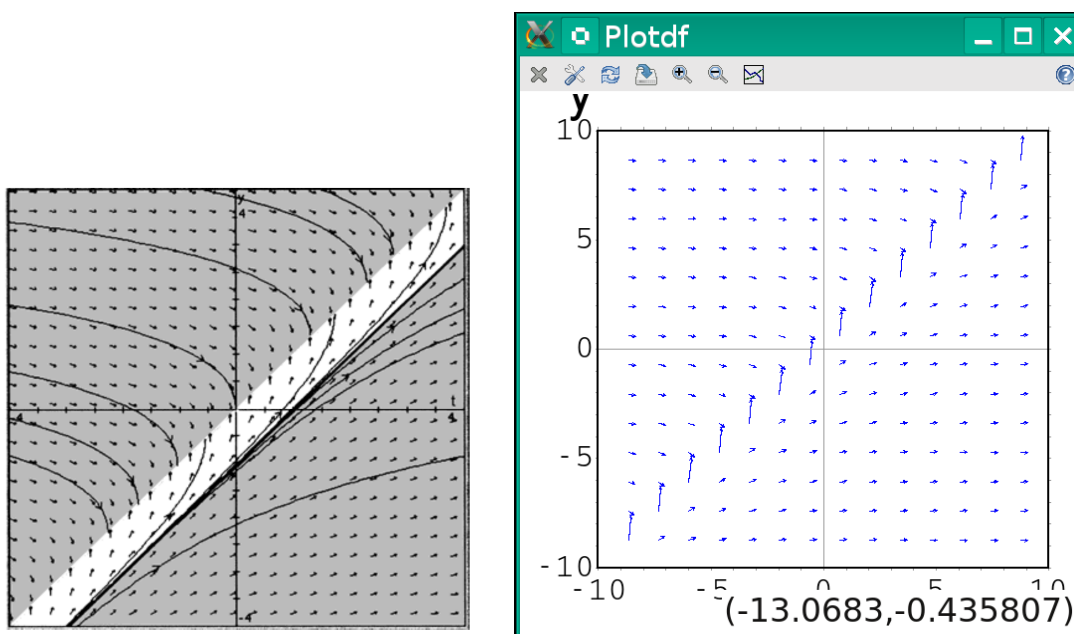


Figure 2: Two versions of the slope field for the ODE 2

Clearly from the plots as $t \rightarrow -\infty$, the solutions come from ∞ (i.e. from very large positive y values) if they start above $y = t$ and from $-\infty$ (i.e. from very large negative y values) if they start (the initial condition is specified) below the line $y = t$. What about the behavior as $t \rightarrow \infty$? Below the line $y = t - 1$, the solutions approach ∞ . Above the line $y = t - 1$, the solutions approach the line $y = t$ at an almost vertical direction. Note that $y = t - 1$ is an asymptote. That's because, for $t \rightarrow -\infty$ and below the line $y = t$, the solution slopes are almost tangent to the line, as can be seen from the plot.

Notes on Word Problems

Sergey V.

①

Ex (2.4.22)

Newton's Law of Cooling (or Heating)

$$\frac{dT}{dt} = k(M-T) ; T(0) = T_0$$

$$\Rightarrow T'(t) + kT = kM \Rightarrow u(t) = e^{kt}$$

$$\frac{d}{dt} [u(t)T(t)] = u(t)kM = kMe^{kt}$$

$$u(t)T(t) = kM \frac{1}{k} e^{kt} + C$$

$$\Rightarrow T(t) = M + Ce^{-kt} = M + (T_0 - M)e^{-kt}$$

At 1 PM: $T_0 = 70$ (soda temp)
 $M = 40$ (fridge temp)

At 1:15 PM: $T(15) = 60$

$$T(t) = 40 + 30e^{-kt} \Rightarrow T(15) = 40 + 30e^{-15k} = 60$$

$$\Rightarrow 30e^{-15k} = 20 \Rightarrow e^{-15k} = \frac{2}{3} \Rightarrow -15k = \ln\left(\frac{2}{3}\right)$$

$$\Rightarrow k = -\frac{1}{15} \ln\left(\frac{2}{3}\right)$$

(the constant k is intrinsic to the object-soda-can - and will not change)

So in the fridge:

$$T(t) = 40 + 30e^{-kt} \quad \text{with } k = -\frac{1}{15} \ln\left(\frac{2}{3}\right) \approx 0.027$$

At some later time (call this time t_r), the soda is moved from fridge back to room (room is at same temperature $\rightarrow 70^\circ\text{F}$). At $t=t_r$, $T(t_r) = 40 + 30e^{-kt_r}$.

The temperature of the soda can will now satisfy a new equation:

$$\frac{dN}{dt} = k(M - N) ; N(0) = T(t_r)$$

initial temp is temp of soda can as its taken out of fridge

$$\Rightarrow N(t) = M + (N_0 - M)e^{-kt} = 70 + (N(0) - 70)e^{-kt}$$

$$\text{But } N(0) = T(t_r) = 40 + 30e^{-kt_r}$$

$$\begin{aligned} \Rightarrow N(t) &= 70 + (30e^{-kt_r} - 30)e^{-kt} \\ &= 70 + 30(e^{-kt_r} - 1)e^{-kt} \end{aligned}$$

Now we know that at 2 PM, the soda temp goes to 60°F . Now t_r is time after 1 pm. So in minutes, time since can was removed from fridge to 2pm is $(60 - t_r)$. So $N(60 - t_r) = 60$

$$\Rightarrow 60 = N(60 - t_r) = 70 + 30(e^{-\kappa t_r} - 1) e^{-\kappa(60 - t_r)} \quad (2)$$

$$\begin{aligned} \Rightarrow -10 &= 30(e^{-\kappa t_r} - 1) e^{-60\kappa + \kappa t_r} \\ &= 30(e^{-\kappa t_r} - 1) e^{-60\kappa} e^{\kappa t_r} \\ &= 30 e^{-60\kappa} - 30 e^{-60\kappa} e^{\kappa t_r} \\ &= 30 e^{-60\kappa} (1 - e^{\kappa t_r}) \end{aligned}$$

$$\Rightarrow -\frac{1}{3} = e^{-60\kappa} (1 - e^{\kappa t_r}) = \alpha (1 - e^{\kappa t_r})$$

where $\alpha = e^{-60\kappa}$ (which we can evaluate since we know κ)

$$\Rightarrow 1 - e^{\kappa t_r} = \frac{-1}{3\alpha} \Rightarrow e^{\kappa t_r} = 1 + \frac{1}{3\alpha}$$

$$\kappa t_r = \ln\left(1 + \frac{1}{3\alpha}\right) \Rightarrow t_r = \frac{1}{\kappa} \ln\left(1 + \frac{1}{3\alpha}\right)$$

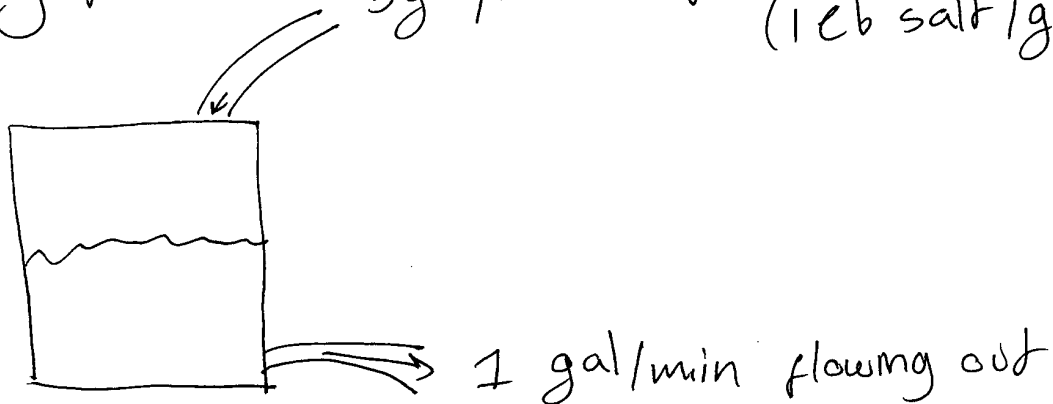
we know κ and α so we can calculate t_r , where t_r is time in minutes after 4 pm at which soda can is removed.

So that:

$$V_p(t) = g(t)e^{-10t} = t - \frac{1}{10}$$

$$\Rightarrow V(t) = V_h(t) + V_p(t) = Ce^{-10t} + t - \frac{1}{10}$$

Ex] (mixing problem) 3 gal/min of salt water (1 lb salt/gal)



A 600 ~~lb~~ gallon tank is filled with 300 gal of pure water. A salt solution (1 lb salt/gal) is flowing in at a rate of 3 gal/min. A drain is opened at bottom allowing solution to leave at rate of 1 gal/min. What will be the salt content when the volume of solution in tank reaches 600 gal?

Let $x(t)$ = amt of salt in tank at time t

$$\frac{dx}{dt} = (\text{rate of salt in}) - (\text{rate of salt out})$$

(2)

$$\frac{dx}{dt} = \underbrace{\frac{1 \text{ lb}}{\text{gal}}}_{\text{concent.}} \cdot \underbrace{\frac{3 \text{ gal}}{\text{min}}}_{\text{inflow}} +$$

$$(-) \frac{x \text{ lb}}{\underbrace{(300 + (3-1)t) \text{ gal}}_{\text{concentration} = \frac{x(t)}{w(t)}}} \cdot \underbrace{4 \frac{\text{gal}}{\text{min}}}_{\text{outflow}}$$

Notice: liquid content $w(t)$ satisfies:

$$\frac{dw}{dt} = (3-1) = 2 ; w(0) = 300$$

$$\Rightarrow w(t) = 2t + C \Rightarrow w(t) = 2t + 300$$

hvs, $\frac{dx}{dt} = 3 - \frac{x}{300+2t} ; x(0) = 0$

$$\frac{dx}{dt} + \frac{1}{(300+2t)} x = 3 \quad \Leftarrow \text{solve by integrating factors}$$

$$\mu(t) = e^{\int \frac{1}{(300+2t)} dt} = e^{\frac{1}{2} \ln |300+2t|} = (300+2t)^{\frac{1}{2}}$$

take $\mu(t) = (300+2t)^{\frac{1}{2}}$ (drop abs value)

$$\Rightarrow \frac{d}{dt} \left[(300+2t)^{\frac{1}{2}} x(t) \right] = 3(300+2t)^{\frac{1}{2}}$$

$$\Rightarrow (300+2t)^{\frac{1}{2}} x(t) = \int 3(300+2t)^{\frac{1}{2}} dt$$

$$= \frac{3}{\frac{1}{2}} \frac{(300+2t)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$\Rightarrow x(t) = (300+2t) + C(300+2t)^{-\frac{1}{2}}$$

$$x(0) = 300 + C(300)^{-\frac{1}{2}} = 0$$

$$\frac{C}{\sqrt{300}} = -300 \Rightarrow C = -300\sqrt{300} = -3000\sqrt{3}$$

$$\Rightarrow x(t) = (300+2t) - 3000\sqrt{3} (300+2t)^{-\frac{1}{2}}$$

tank is full (600 gal) when $w(t) = 600$

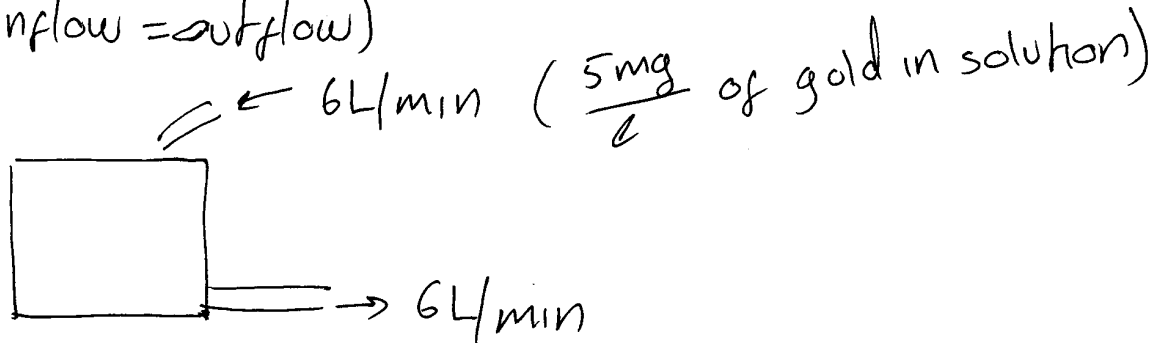
$$\text{recall: } w(t_f) = 2t_f + 300 = 600 \Rightarrow 2t_f = 300 \Rightarrow t_f = 150$$

$$\text{Thus, } x(t_f) = (300 + 300) - 3000\sqrt{3} (600)^{-\frac{1}{2}} \approx 388 \text{ lb of salt}$$

Notice: here we really had a system of ODEs.

$$\frac{dw}{dt} = 2; \quad \frac{dx}{dt} = 3 - \frac{x(t)}{w(t)} \quad \left| \quad \begin{array}{l} w(0) = 300 \\ x(0) = 0 \end{array} \right.$$

Ex] (inflow = outflow)



At $t=0$, gold at 5mg/L flows in. Initially 1000L of water. What is the IVP for amount of gold at time t ?

Notice: $\frac{dw}{dt} = 6 - 6 = 0 \Rightarrow w(t) = 1000$ for all t .
 $w(0) = 1000$

$x(t)$ = amt of gold at time t ; $x(0) = 0$

$$\Rightarrow \frac{dx}{dt} = \underbrace{\frac{6\text{L}}{\text{min}} \cdot \frac{5\text{mg}}{\text{L}}}_{\text{in}} - \frac{6\text{L/min}}{1000\text{L}} \cdot \frac{x(t)\text{mg}}{w(t)}$$

$$= 30\text{mg/min} - 6 \frac{x}{1000}\text{mg/min}$$

$$\Rightarrow \frac{dx}{dt} = 30 - \frac{3}{500}x \quad ; \quad x(0) = 0$$

solve by integrating factors
 or variation of params

$$: x(t) = 5000 \left(1 - e^{-\frac{3}{500}t} \right)$$

Notes on Simple Numerical Methods for First Order ODEs

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October 6, 2015

1 Introduction

The solutions to many ODEs can not be obtained in analytical form. Even if the ODE in question can be solved, the solution is often written in terms of an integral operator or in implicit form. For this reason, numerical algorithms to obtain approximate solutions to the ODEs are very important, as they are used in many (if not most) of real world applications involving differential equations. The numerical methods are typically applicable to the IVP (that is to the ODE with an initial condition). For first order differential equations, the IVP takes the form:

$$\frac{dy}{dt} = f(t, y) \quad ; \quad y(t_0) = y_0$$

where t is often referred to as the time variable (depending on the equation, this may have physical significance). In practice, we can use x instead of t or any other variable name.

2 Euler's Method

Euler's method is based on a simple tangent line approximation. That is, suppose we start from the point (t_0, y_0) which we know, since both t_0 and the value of $y(t)$ at $t = t_0$ (y_0), are specified by the initial condition. We may not know the true solution $y(t)$ emanating from that point, but we do know the slope of the solution at that point. That is, the slope $y'(t)$ is given by $f(t, y)$ for all of t . The equation for the tangent line passing through (t_0, y_0) is:

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

We can use this as an approximation to our solution $y(t)$. Then at a certain time $t = t_1$, the approximate solution is

$$y_1 = y_h(t_1) = y_0 + f(t_0, y_0)(t_1 - t_0)$$

Note that we only expect this to be accurate for t_1 close to t_0 . Then at $t = t_1$, we can write the equation for the new tangent line:

$$y = y_1 + f(t_1, y_1)(t - t_1)$$

We then use the new tangent line to infer the approximate value of y at $t = t_2$, then:

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

Then we can continue in the same fashion getting approximate solutions at $t = t_3, t_4, \dots, t_N$ by constructing successive tangent lines. Note that the spacing between the successive times t is usually chosen to be uniform (although it can also be non-uniform in more advanced implementations). That is, we set $t_{k+1} - t_k = h$ where h is usually called the time step parameter. If the spacing is uniform, we have that $h = \frac{t_N - t_0}{N}$. Below, is an illustration of the Euler method approximation:

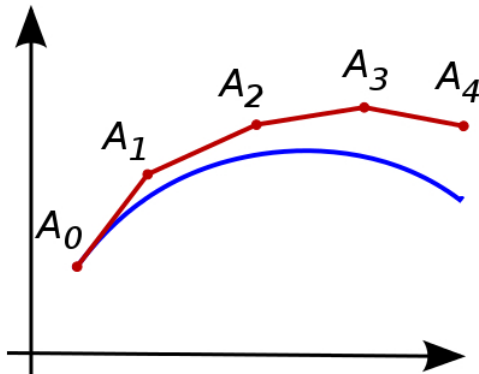


Figure 1: Illustration of Euler's method: tangent line approximation (red) and actual solution (blue).

Note that based on the above discussion, if we split the time interval $(t_N - t_0)$ into N different parts, then Euler's method can be implemented simply as:

$$y_{n+1} = y_n + hf_n \quad ; \quad t_{n+1} = t_n + h \quad \text{for } n = 0, 1, 2, \dots, (N - 1)$$

where $y_0 = y(t_0)$ and $f_n = f(t_n, y_n)$. As an example consider the problem $y' + 2y = x^3 e^{-2x}$ with $y(0) = 1$. Note the independent variable is x here, not t . We like to approximate y at $x = 0.1$ and $x = 0.2$. If we take $h = 0.1$, then we have that $x_0 = 0, x_1 = 0.1, x_2 = 0.2$ (and so, $h = \frac{0.2-0}{2} = 0.1$). It follows that: $y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1 * f(0, 1) = 1 + 0.1 * (-2) = 0.8$ and $y_2 = y_1 + hf(x_1, y_1) = 0.8 + 0.1 * f(0.1, 0.8) = 0.8 + 0.1 * (-2 * (0.8) + (0.1)^3 e^{-0.2}) \approx 0.64$.

We can also program this algorithm. Using Matlab (or Octave) we can write a simple code to implement the algorithm for a given IVP. Consider for example, the following first order ODE IVP:

$$\frac{dy}{dt} = K(y - \alpha) \quad ; \quad y(0) = y_0$$

where K and α are constants. The analytical solution is:

$$y(t) = y_0 e^{Kt} + \alpha(1 - e^{Kt})$$

To derive this we used separation of variables and substitution $u = y - \alpha$:

$$\frac{dy}{K(y - \alpha)} = dt \implies \frac{du}{u} = K dt \implies \log|u| = Kt + C_1 \implies u(t) = C e^{Kt} \implies (y(t) - \alpha) = C e^{Kt} \implies y(t) = \alpha + C e^{Kt}$$

Plugging in the IC, we get: $y(0) = \alpha + C = y_0 \implies C = y_0 - \alpha$, which gives the solution $y(t) = \alpha + (y_0 - \alpha)e^{Kt}$ we quote.

We will compare the true solution to one obtained with Euler's method. To do so, create two files. One called `y_true.m`, which is a simple function for getting back the true solution value for a given value of t :

```
1 % true solution to IVP
2 function val = y_true(t,y0,K,alpha)
3     val = y0*exp(K*t) + alpha*(1 - exp(K*t));
4 end
```

and another file called `run_euler.m` which actually implements Euler's method and compares to the true solution. Notice that we must set the initial and final times t . Obviously if we want to construct the approximate solution from $t = 0$ to a large t , more iterations are required and the accuracy and efficiency of the method becomes important.

```
1 % euler's method for dy/dt = K*(y-alpha); y(0) = y0
2 K = 1;
3 alpha = 20;
4 y0 = 100;
5
6 tinit = 0;
7 tfinal = 10;
8 N = 50; % how many time intervals to use
9 h = (tfinal - tinit)/N;
10
11 % set y to be all zero at first
12 y = zeros(N,1);
13 t = zeros(N,1);
14
15 % record the initial condition
16 y(1) = y0;
17 t(1) = tinit;
18
19 % loop eulers algorithm
20 for n=1:(N-1)
21     fn = K*(y(n)-alpha);
22     y(n+1) = y(n) + h*fn;
23     t(n+1) = t(n) + h;
24 end
25
26 figure(1);
27 hold on;
28 plot(t,y_true(t,y0,K,alpha),'r','linewidth',3);
```

```

29 plot(t,y,'b','linewidth',3);
30 legend('TRUE','EULER');
31 hold off;
32 print('-depsc','images/euler_output1.eps');

```

Copy all files to one directory and make the subdirectory called images. Open matlab and run the file run_euler.m. Then refine the N parameter to 500 from 50 and run again. You will see the output in Figure 2 below. Notice that as N is increased, the step size h decreases (that is, becomes finer) and the approximate solution obtained with Euler's method becomes more accurate. However, Euler's method is a simple method known as a first order method and is often unable to accurately model solutions to more complex equations for a given value of h .

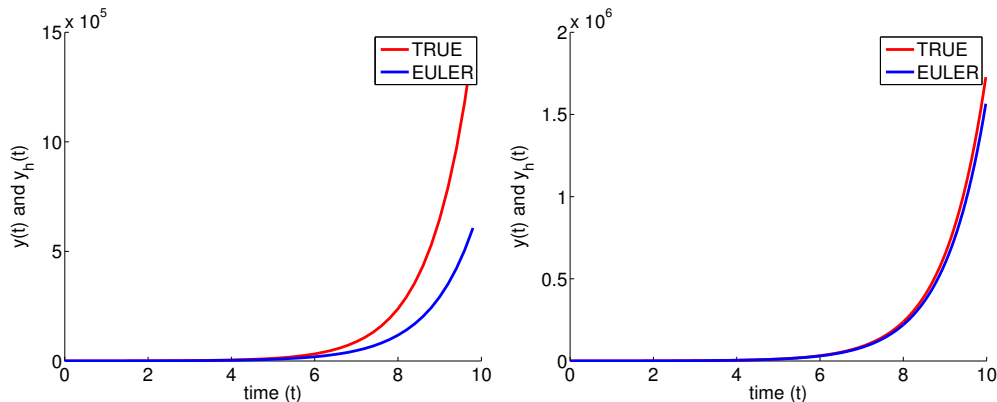


Figure 2: Output with $N = 50$ and $N = 500$. Notice the improved accuracy of the approximation as the step size is reduced.

2.1 Error Analysis

Consider the following expansion at the first step. We have $y(t_0) = y_0$ and want to compute $y(t_1) = y(t_0 + h)$:

$$y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + \mathcal{O}(h^3)$$

We know that $y'(t_0) = \frac{dy}{dt}|_{t=t_0} = f(t_0, y_0)$. Thus, if we only use two terms of this series this leads us to:

$$y(t_0 + h) \approx y(t_0) + hf(t_0, y_0) = y_1$$

which corresponds to one step of Euler's method: $y_{n+1} = y_n + hf_n$ (first step corresponds to $n+1 = 1$). This means that the error made by one step of Euler's method (known as local truncation error) is:

$$y(t_1) - y_1 = y(t_0 + h) - [y(t_0) + hf(t_0, y_0)] = \frac{h^2}{2}y''(t_0) + \mathcal{O}(h^3) = \mathcal{O}(h^2)$$

It can be shown that after n steps of Euler's method, the global truncation error is: $|y_n - y(t_n)| \leq Ch$ for some constant C . Note that when h is small (i.e. $0 < h < 1$, we have $h > h^2 > h^3 > \dots$).

Smaller step size h corresponds to less error. However, due to roundoff errors, it is bad to take h too small. This is because a very small h will correspond to a lot of Euler steps (large N), when going from t_0 to some final time t_f . If at each step we have a roundoff error (due to finite precision arithmetic), we will eventually accumulate a lot of error. Thus there is a tradeoff and a certain small enough (but not too small) time step h which works well, which is problem dependent.

3 Runge-Kutta Methods

We now discuss Runge-Kutta methods, which are more accurate than the simple Euler scheme. Consider the following expansion:

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \mathcal{O}(h^3)$$

As before, if we only use two terms of this series, this leads us to:

$$y(t+h) \approx y(t) + hf(t, y)$$

which corresponds to Euler's method: $y_{n+1} = y_n + hf_n$. Next, suppose we take also the third term in the Taylor expansion. Then:

$$y''(t) = \left(\frac{\partial}{\partial t}f(t, y)\right)\frac{dt}{dt} + \left(\frac{\partial}{\partial y}f(t, y)\right)\frac{dy}{dt} = f_t(t, y) + f_y(t, y)f(t, y)$$

Thus, we get:

$$\begin{aligned} y(t+h) &= y(t) + hy'(t) + \frac{h^2}{2}(f_t(t, y) + f_y(t, y)f(t, y)) + \mathcal{O}(h^3) \\ &= y(t) + hf(t, y) + h\left(\frac{h}{2}f_t(t, y) + \frac{h}{2}f_y(t, y)f(t, y)\right) + \mathcal{O}(h^3) \end{aligned}$$

Again, we can replace $y'(t)$ by $f(t, y)$ but now we need to also replace the term in the brackets $\left(\frac{h}{2}f_t(t, y) + \frac{h}{2}f_y(t, y)f(t, y)\right)$ with quantities which we can compute. To do this, consider the following multivariate Taylor expansion:

$$f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right) = f(t, y) + \frac{h}{2}f_t(t, y) + \frac{h}{2}f_y(t, y)f_y(t, y) + \mathcal{O}(h^2)$$

Which implies: $\frac{h}{2}f_t(t, y) + \frac{h}{2}f_y(t, y)f_y(t, y) = f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right) - f(t, y) + \mathcal{O}(h^2)$. Thus:

$$\begin{aligned} y(t+h) &= y(t) + hf(t, y) + h\left(f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right) - f(t, y)\right) + \mathcal{O}(h^2) \\ &= y(t) + hf\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right) + \mathcal{O}(h^2) \end{aligned}$$

Based on the above analysis, a second order Runge-Kutta scheme (often called RK2) for the IVP

$$\frac{dy}{dt} = f(t, y) \quad ; \quad y(t_0) = y_0$$

proceeds as follows:

$$y_{n+1} = y_n + hk_{n2} \quad ; \quad t_{n+1} = t_n + h$$

$$\text{where } k_{n1} = f(t_n, y_n) \text{ and } k_{n2} = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_{n1}\right)$$

Note that this method is not unique. That is, there are other versions of Runge Kutta order 2 schemes, based on different multivariate expansions for replacing $\left(\frac{h}{2}f_t(t, y) + \frac{h}{2}f_y(t, y)f(t, y)\right)$. Try to implement this method for the above Logistic equation IVP and compare to Euler's method for the same value of step size h . Since this method is based on the first three Taylor series terms, it is expected to be more accurate for the same h than Euler's scheme. For more information on how the Runge-Kutta order 2 scheme is derived, please see: http://math.iit.edu/~fass/478578_Chapter_3.pdf and http://mathforcollege.com/nm/mws/gen/08ode/mws_gen_ode_txt_runge2nd.pdf.