Covered Material and Resources for Part2 (Exam 1 - Exam 2)

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1 Systems of ODEs and Nullclines

Textbook: Section 2.6

Introduction to systems of ODEs. Simple examples come from mixing problems - a tank contains some water, a solution containing salt flows in through one pipe and the mixed solution flows out of the tank through a different pipe. There are two variables: the amount of liquid in the tank $W(t)$ and the amount of salt in tank $x(t)$. If the flow in doesn't equal to flow out, then both change with time resulting in a system of two differential equations. If flow rate in is same as flow rate out then only $x(t)$ changes with time, as $W'(t) = 0$ in that case.

When we have a system of ODEs:

$$\frac{dx}{dt} = f(x,y) \quad \text{and} \quad \frac{dy}{dt} = g(x,y)$$

then the method of Nullclines can be used to analyze equilibrium points and plot simple behavior of the solution curves given in the $x - y$ plane at each time t by the vector $(x(t), y(t))$. Note that the slope of the solution curve at any given point of the $x - y$ plane is the vector $x'(t)\hat{\imath} + y'(t)\hat{\jmath} = f(x(t), y(t))\hat{\imath} + g(x(t), y(t))\hat{\jmath}$. In particular, when $f(x,y) = 0$, we have that the slope of the solution is vertical (it is given by $g(x,y)\hat{\jmath}$). Thus, the set of points (x,y) satisfying $\frac{dx}{dt} = f(x,y) = 0$ is known as the the vertical (or v) nullclineand has slope $\vec{s}_v = g(x,y)\hat{\jmath}$ (pointing down or up). The set of points (x,y) satisfying $\frac{dy}{dt} = g(x,y) = 0$ is known as the horizontal (or h) nullcline and has slope $\vec{s}_h = h(x,y)\hat{\imath}$ (pointing left or right). Points (x,y) for which $f(x,y) = 0 = g(x,y)$ are equilibrium points of the ODE system. A simple example is:

$$\begin{aligned}\frac{dx}{dt} &= 1 - x - y = f(x,y) \\ \frac{dy}{dt} &= 1 - x^2 - y^2 = g(x,y)\end{aligned}$$

Then the horizontal nullcline is the set of points $g(x,y) = 0 \implies x^2 + y^2 = 1$, a circle of radius 1. The vertical nullcline is the set of points $f(x,y) = 0 \implies x + y = 1$, which is a line. Equilibrium points are $(1,0)$ and $(0,1)$. To determine if the equilibrium points are stable or unstable we must plot the directions of the slopes along the horizontal and vertical nullclines. Along the horizontal nullcline the slopes are pointing left or right. To determine direction at any given point note that the slope direction is given by the vector $\frac{dx}{dt}\hat{i}$ and hence depends on the sign of $\frac{dx}{dt} = f(x, y)$ (positive sign corresponds to the right direction, negative to the left). Similarly, along the vertical nullcline the slopes point either up or down. The slope has the direction of the vector $\frac{dy}{dt}\hat{j}$ and so depends on the sign of $g(x, y)$ (positive sign corresponds to up, negative to down). From this we can determine that $(0, 1)$ is stable (solution curves around the point do not go away from the point) and $(1, 0)$ unstable (solution curves around the point go away from the point in some directions) equilibrium. See handwritten examples at the end of the pdf.

Other references:

- mcb.berkeley.edu/courses/mcb137/exercises/Nullclines.pdf, http://www.sosmath.com/diffeq/system/qualitative/qualitative.html

2 Vectors and Matrices

Textbook: Section 3.1,3.2,3.3 (matrices and basic linear algebra, system of equations, RREF, matrix inverses)

Vectors are written as $b \in \mathbb{R}^m$, which denotes a column or row vector of m rows (usually, column vector by default). For a row vector, one would write b^T. Example:

$$b = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \text{and} \quad b^T = \begin{bmatrix} \alpha & \beta \end{bmatrix}$$

We write real matrices using similar notation: $A \in \mathbb{R}^{m \times n}$ denotes a real valued matrix of m rows and n columns. Basic operations with matrices: addition, scalar multiplication, matrix-matrix and matrix vector multiplication. Note that matrix-matrix multiplication is defined as $C = AB$ for matrices $A \in \mathbb{R}^{m \times r}$ and $B \in \mathbb{R}^{r \times n}$ resulting in $C \in \mathbb{R}^{m \times n}$. Matrix multiplication is not defined for matrices A and B for which the number of columns of A is different from the number of rows of B. If the conditions are satisfied then C_{ij} (the element of C in row i and column j) is given by:

$$C_{ij} = \sum_{k=1}^{r} A_{ik} B_{kj}$$

For each entry (i, j), this corresponds to taking row vector i of A and dotting it with the column vector j of B. For matrix-vector multiplication, we must have that $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$ where x is a column vector of n elements. Then $b = Ax$ is defined and:

$$b_i = \sum_{k=1}^{n} A_{ik} x_k$$

The transpose of a matrix is always defined. If A has elements a_{ij} then A^T has elements a_{ji} (the column vectors of A become the row vectors of A^T and vice versa). Note that if the matrix matrix product AB is defined, then it can be proved using componentwise notation and the formula for matrix-matrix multiplication that $(AB)^T = B^T A^T$. A particular special matrix is called the identity matrix. This matrix is diagonal (has nonzero elements only along the diagonal). For example, a 2×2 identity matrix I_2 is:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies Ix = x \quad \text{and} \quad IA = AI = A$$

for any $m \times n$ matrix A when I is $n \times n$. Introduction to the range (or column space) of a matrix. For a matrix $A \in \mathbb{R}^{m \times n}$ the range is the set of vectors Ax for all possible vectors $x \in \mathbb{R}^n$. For example for the matrix:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

the range is the set of all vectors $\{Ax | x \in \mathbb{R}^2\}$ of the form:

$$\begin{bmatrix} C \\ 0 \end{bmatrix}$$

for $C \in \mathbb{R}$. Since, for any $x \in \mathbb{R}^2$, we have:

$$Ax = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 0 \end{bmatrix}$$

This means the range is only a subset (in fact, a subspace) of $\mathbb{R}^2$.

Linear systems of equations using matrix methods: a set of linear equations can have a unique solution, infinitely many solutions, or no solution. We write down a system of equations in matrix form $Ax = b$ using the augmented matrix construction $[A|b]$. Consider for example the system of equations:

$$\begin{aligned} x + y &= 4 \\ x - y &= 0 \end{aligned}$$

Then we can write:

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

by performing elementary row operations on the matrix (replacing a row by a sum or difference of the current row and a scalar multiple of another row, scaling a row, switching rows) and reducing the matrix to reduced row echelon form (RREF). In this case, we see that the system of equations has a unique solution $x = y = 2$. Row reduced echelon form is characterized as follows:

- All zero rows are at the bottom.
- The leading coefficient (first nonzero number from the left) of a nonzero row is a 1 and always strictly to the right of the leading coefficient of the row above it.
- Every leading coefficient (the pivot element 1) is the only nonzero entry in its column.

Note: if only the first two requirements are satisfied, matrix is said to be in row echelon form. As an another example consider the reduction sequence:

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that the last matrix R obtained from A via elementary row operations is in reduced row echelon form (note the pivot columns have pivots as 1 and these are the only nonzeros in the respective columns). Consider now to find the nullspace of the matrix above (the set of solutions to $Ax = 0$). We have the reduction:

$$\begin{bmatrix} 1 & 2 & 1 & | & 0 \\ -2 & -3 & 1 & | & 0 \\ 3 & 5 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

which implies $x_1 - 5x_3 = 0$ and $x_2 + 3x_3 = 0$. If we set $x_3 = \alpha$, we get $x_1 = 5\alpha$ and $x_2 = -3\alpha$. So that the set of solutions to $Ax = 0$ is given by:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5\alpha \\ -3\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \quad \text{for} \quad \alpha \in \mathbb{R}$$

This means the basis for the nullspace of this A is one dimensional, consisting of the vector above and the nullspace itself is defined as the span of this vector:

$$\mathbf{null}(A) = \mathbf{span}\left(\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \right)$$

In cases, where the system has infinitely many solutions or no solutions reduction to form where the left part of the augmented matrix (everything other than the last, rightmost column) is the identity matrix as above is not possible. The process of solving linear systems by transforming the augmented matrix is known as Gaussian Elimination.

Superposition principle. The map $L(x) = Ax$ is linear for any matrix $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^{n\times 1}$. This is easy to verify: $L(C_1x + C_2y) = C_1L(x) + C_2L(y)$. Thus, a solution to the system of equations $L(x) = b$ (same as $Ax = b$) is given as the sum of the solution to the homogeneous problem and a particular solution $x = x_h + x_p$, where x_h satisfies $Ax_h = 0$. The homogeneous solution can be obtained by forming the augmented matrix $(A|0)$ (i.e. last column is all zeros) and obtaining the row reduced form. A particular solution is any solution to the system. It can be obtained by setting arbitrary values for some variables as long as the non-homogeneous system of equations remains satisfied. Note that in the case when $Ax = b$ has infinitely many solutions, it is possible to write the same solution family many different ways.

Matrix inverses. For square matrices $A \in \mathbb{R}^{n\times n}$ a matrix inverse A^{-1} sometimes exists. It exists if and only if $det(A) \neq 0$. The inverse satisfies $AA^{-1} = A^{-1}A = I_n$ where I_n is the identity matrix (usually denoted just by I, irrespective of size). Note that in contrast to this, typically $AB \neq BA$

even if both operations are defined. If the inverse matrix exists, the solution to the system $Ax = b$ is unique: it is given by $x = A^{-1}b$. Again, this only holds for square matrices. Another consequence of the inverse matrix existing is that the only solution to the homogeneous equations $Ax = 0$ is $x = A^{-1}0 = 0$, which is the same as saying that the null space of the matrix (which is the set $\{x : Ax = 0\}$) is trivial, consisting only of the zero vector. If the inverse matrix does not exist then the null space contains other vectors besides the zero vector. Inverse matrices can be found using row reduction. By reducing the matrix $[A|I]$ to $[I|A^{-1}]$. For example, consider the reduction below to find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Notice that $det(A) \neq 0$ so the inverse matrix A^{-1} exists (it's useful to check this before carrying out the row reduction work). We obtain:

$$\begin{aligned} [A|I] \quad &= \begin{bmatrix} 1 & 0 & 1 & |1 & 0 & 0 \\ 1 & -1 & 1 & |0 & 1 & 0 \\ 1 & -1 & 2 & |0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & |1 & 0 & 0 \\ 0 & -1 & 0 & |-1 & 1 & 0 \\ 1 & -1 & 2 & |0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & |1 & 0 & 0 \\ 0 & -1 & 0 & |-1 & 1 & 0 \\ 0 & -1 & 1 & |-1 & 0 & 1 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 0 & 1 & |1 & 0 & 0 \\ 0 & 1 & 0 & |1 & -1 & 0 \\ 0 & -1 & 1 & |-1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & |1 & 0 & 0 \\ 0 & 1 & 0 & |1 & -1 & 0 \\ 0 & 0 & 1 & |0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & |1 & 1 & -1 \\ 0 & 1 & 0 & |1 & -1 & 0 \\ 0 & 0 & 1 & |0 & -1 & 1 \end{bmatrix} \end{aligned}$$

Hence, we conclude that for:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad \text{the inverse matrix is:} \quad A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Determinants are defined also for square matrices (which makes sense as they can be used to test if an inverse matrix exists or not). The determinant of a matrix is a scalar $det(A) = d \in \mathbb{R}$ and can be negative, positive or zero. The determinant can be computed by expanding along any row or column of the matrix. Often, it greatly simplifies the computation to select the right row or column to expand along. A good example is an upper triangular matrix. The formula for the determinant of an $n \times n$ matrix is:

$$|A| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}|M_{ij}|$$

where the expansion is done along the i-th row, or:

$$|A| = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}|M_{ij}|$$

where the expansion is done along the j-th column. The term $|M_{ij}|$ denotes the determinant of the so called minor matrix obtained by deleting the i-th row and j-th column of A. Determinants satisfy several properties: for example, $|AB| = |A||B|$ and $|A^{-1}| = \frac{1}{|A|}$ (notice that if A^{-1} exists then $|A| \neq 0$

in previous fraction). To show the previous equality, note that $|AA^{-1}| = |I| = |A||A^{-1}|$. Since $|I| = 1$, $|A^{-1}| = \frac{1}{|A|}$ follows.

The determinant can be used to obtain solutions to square linear systems $Ax = b$ (where A is square matrix) and A^{-1} exists. In this case, the components of the unique solution x can be obtained via Cramer's rule. The advantage of this approach is that it does not involve the computation of the inverse matrix or any kind of RREF procedure, just computations of determinants.

Other references:

- pdfs https://math.dartmouth.edu/archive/m23s06/public_html/handouts/row_reduction_examples.pdf, http://www.sosmath.com/matrix/system1/system1.html
- determinants https://www.math.drexel.edu/~jwd25/LM_SPRING_07/lectures/lecture4B.html
- http://www.millersville.edu/~bikenaga/linear-algebra/matrix/matrix.html
- Diffeq for Engineers book (Matrices and Linear Algebra Section) http://www.jirka.org/diffyqs/diffyqs.pdf

3 Elementary Row Operations, Rank, Vector Spaces

Textbook: Section 3.4,3.5 (review of linear systems and inverses, rank of a matrix, vector spaces)

Recall the three allowed row operations used to get a matrix into RREF form:

- (A) Interchange two rows: $R_i \longleftrightarrow R_j$.
- (B) Scale a row by a constant: $R_i^* \leftarrow kR_i$ for $k \in \mathbb{R}$.
- (C) Replace a row by the sum of the current row plus a scalar multiple of a different row: $R_i^* \leftarrow R_i + kR_j$ for $k \in \mathbb{R}$ and $i \neq j$.

It turns out that these three operations can be performed via matrix matrix multiplication with elementary matrices. For $A \in \mathbb{R}^{3\times3}$, the following transformation matrices can be used:

$$E_{\text{int}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad E_{\text{scale}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad E_{\text{repl}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$$

We apply these matrices from the left forming $B = EA$. The first matrix acts to interchange rows 1 and 2. The second matrix scales the second row by k and the third matrix replaces the third row by the sum of k times the first row and the original third row. Thus, any matrix A can be turned into R where $R = E_k E_{k-1} \dots E_1 A$ (via a sequence of products of elementary matrices).

The concept of matrix rank is important to judge the type of solutions to a linear system $Ax = b$ with a general matrix $A \in \mathbb{R}^{m\times n}$ (not necessarily square). We can have either no solution, a unique solution or infinitely many solutions. The rank of a matrix M is max number of linearly independent column vectors and is also equal to the max number of linearly independent row vectors (i.e. rank of M and of M^T is the same and is always $\leq \min(m,n)$). For example, the following matrix has rank 2:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 2 \\ 3 & 9 & 4 \end{bmatrix}$$

We can determine this two ways: reduce to RREF (or at least to row echelon form) and find the existence of two pivot columns, or simply observe above that column vectors one and two are multiples of each other and the third column cannot be written as a linear combination of the first two. In general, for larger matrices, to determine rank, row reduction is necessary and rank is equal to the number of pivot columns in rref(A). An $m \times n$ matrix can have rank at most $\min(m,n)$. A matrix with all zero entries has zero rank. Otherwise the rank is at least one. We have the following results regarding solutions. We take the system and form the augmented matrix $(A|b)$ (i.e. add vector b as last column). Notice that if $x = (x_1, \ldots, x_n)^T$, then $A = [a_1, \ldots, a_n]$ and $Ax = x_1a_1 + \cdots + x_na_n$ (a linear combination of the columns of A). Hence x is a solution if and only if b is a linear combination of the columns of A (i.e. $b = x_1a_1 + \cdots + x_na_n$ so b is linearly dependent with the column vectors of A). Then:

- If, $rank(A) < rank(A|b)$ then there is no solution to $Ax = b$ (this relationship between the ranks implies that b is linearly independent with the column vectors of A and cannot be written as a linear combination of them).
- If, $rank(A) = rank(A|b) = n$, then there is a unique solution.
- If, $rank(A) = rank(A|b) < n$, there are many solutions to the system.

Note that for squared systems, it is enough to check that the determinant of A is nonzero, to conclude that a unique solution exists (as $det(A) \neq 0$ implies the existence of A^{-1}).

Vector Spaces. A vector space $\bar{V}$ is a collection of objects (vectors) which satisfy a number of properties. In a vector space the vectors may be added or subtracted and multiplied by scalars and still remain part of the space. Note that vectors can be functions of t (time dependent) or simply one element long. Thus when we refer to vector spaces, we may actually be referring to collections of functions $f(t)$ or polynomials, not necessarily vectors of several elements. For example, we can make a one to one correspondence between polynomials of up to second degree and 3 variable vectors. The collection of objects is called a vector space if it satisfies a number of properties. Amongst these, the primary properties are as follows. Let us assume that $x, y, z \in \bar{V}$ are elements of the vector space and C and D are scalar constants. Then:

- (1) $x + y \in \bar{V}$ (that is, for any two elements in the space, their sum must be in the space)
- (2) $Cx \in \bar{V}$ (any scalar multiple of an element in the space is also in the space)
- (3) $0 \in \bar{V}$ (there is a zero element in the space)
- (4) $-x \in \bar{V}$ (for any element in the space it's negative is also in the space)

where properties (3) and (4) above are just special cases of (2). Other properties concern basic linearity operations with two elements of the space: The reason we separate the two sets of properties is that

$$(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})$$

$$\vec{x}+\vec{y}=\vec{y}+\vec{x}$$

$$1\vec{x}=\vec{x}$$

$$c(\vec{x}+\vec{y})=c\vec{x}+c\vec{y} \quad ; \quad (c+d)\vec{x}=c\vec{x}+d\vec{x}$$

$$c(d\vec{x})=cd\vec{x}$$

usually $(1)-(2)$ are used to disprove that something is a vector space, while the rest are linearity properties which are easier to see if they are satisfied or not. For example, consider the space of odd degree polynomials. Then $p_1 = x^3 + x^2$ is in the space and $p_2 = -x^3 + 2x^2$ is in the space. But the sum $p_1 + p_2 = 3x^2$ is not in the space. Hence, this space is not a vector space. Consider the space of all pairs of real numbers (x, y) such that $x \geq y$. This is not a vector space because if some element $p \in \bar{V}$ then it is not necessarily true that $-p \in \bar{V}$. For example for $p = (2, 1)$, $-p = (-2, -1)$ and $-p$ is not in the space since $-2 < -1$. Some examples of vector spaces are the space of real vectors $\mathbb{R}^m$ (for any $m \geq 1$), the set of solutions to the linear homogeneous differential equation $L[y] = 0$ where L is a linear differential operator, and the space of polynomials of degree less than or equal to n. Notice for example, that the set of invertible $n \times n$ matrices is not a vector space. If two matrices are invertible (then both have determinant nonzero), but it's possible that their sum $A + B$ has determinant zero, which would make it non-invertible and so the sum may not be in the space. Also, the set of solutions to a nonlinear homogeneous equation $y' + y^2 = 0$ is not a vector space as it fails the linearity properties (i.e. if y_1 and y_2 satisfy the equation, $y_1 + y_2$ will not due to the non-linearity - this is easy to verify by plugging in into the equation). The set of solutions to a linear non-homogeneous equation is also not a vector space for the same reason. However, the set of solutions to an equation that is linear and homogeneous is a vector space.

Other references:

- Pauls Notes http://tutorial.math.lamar.edu/Classes/Alg/AugmentedMatrix.aspx,http://tutorial.math.lamar.edu/Classes/Alg/AugmentedMatrixII.aspx
- Matrix rank www.math.tamu.edu/~fnarc/psfiles/rank2005.pdf
- Vector Spaces and Subspaces http://www.math.niu.edu/~beachy/courses/240/06spring/vectorspace.html
- Wikibook on Linear Algebra http://en.wikibooks.org/wiki/Linear_Algebra/Definition_and_Examples_of_Vector_Spaces

4 Linear independence of vectors and functions, span, basis, dimension

Textbook: Section 3.6 (linear independence, Wronskian, span, basis, dimension)

Linear independence of vectors. A set of vectors $\{v_1, \ldots, v_r\}$ of n elements (i.e. each $v_k \in \mathbb{R}^n$) is linearly independent if and only if the solution to the equation:

$$C_1 v_1 + \cdots + C_r v_r = \vec{0} \tag{4.1}$$

is satisfied only by the solution $C_1 = \cdots = C_r = 0$. This means that no vector in the set can be written as a linear combination of other vectors in the set. Note that this all zero solution will always satisfy the above equation (4.1). If it is the only solution, then the vectors are linearly independent. In not, the vectors are linearly dependent and some vectors in the set may be written as a linear combination of the others. Notice that if we have two vectors v_1 and v_2 which are linearly dependent, then $C_1 v_1 + C_2 v_2 = 0$ for some nonzero C_1 and C_2, which implies that $v_1 = -\frac{C_2}{C_1} v_2$, which means that v_1 is a constant multiple of v_2. So for two vectors it's easy to verify if or not they are linearly independent, just by inspection. If we have r vectors of dimension n (i.e. each vector has n elements) and $r > n$, then at most n of them can be linearly independent, since if we form a matrix out of these vectors the rank is at most $\min(r, n)$. If we have n vectors of n elements each, we can use the determinant test to check independence. We stack the vectors as columns or rows of the matrix and compute the determinant. If it is nonzero, they are linearly independent. In the case of r vectors with n elements where $r < n$, we have to do row reduction. The rank of the resulting matrix is at most $\min(r, n) = r$ in this case, so we have to check the number of pivot columns - if the rank is indeed r, the vectors are linearly independent, otherwise they are linearly dependent.

The span of a set of vector is the set of all possible linear combinations of these vectors. It is equivalent to the range of a matrix whose columns are the vectors in the set. Thus for example, the span of vectors:

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Is all the vectors of the form $C_1 e_1 + C_2 e_2$ for $C_1, C_2 \in \mathbb{R}$:

$$span(e_1, e_2) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \mathbb{R}^2$$

The column space of a matrix is the span of the column vectors of the matrix. That is, a matrix $A \in \mathbb{R}^{m \times n}$ is composed of n column vectors $a_1, \ldots, a_n$ and the column space of A is the span of this set of vectors. It is also known as the range of the matrix A. For $A \in \mathbb{R}^{m \times n}$, the range is the set of all possible $\{Ax\}$ for $x \in \mathbb{R}^n$ (this is the same as the set of all linear combinations of the matrix columns).

Note that an $m \times n$ matrix A can be represented as a set of n column vectors $\{v_1, \ldots, v_m\}$ or m row vectors $\{r_1, \ldots, r_m\}$. The column space or range of A is the span of all the column vectors. While the row space of A is the span of the row vectors. Note that while the reduced matrix R obtained from A via elementary row operations has the same row space as A, it does not have the same column space. Hence, for basis of column space, we must select the linearly independent columns of A, which we usually pick via the pivot columns of R. For basis of row space, we can take the linearly independent rows of A (but care must be taken if switching is done) or those of R (a safer alternative).

For a set of functions $f_1(t), \ldots, f_n(t)$ we can also talk about linear independence and if they are $(n-1)$ times differentiable, we can make use of the concept of the Wronskian determinant. Assume that the functions are all in the space $\mathcal{C}^{n-1}$ (that is $(n-1)$ times continuously differentiable). For linear independence we must have that $C_1 f_1(t) + \cdots + C_n f_n(t) = 0$ is satisfied only for $C_1 = \cdots = C_n = 0$ for all $t \in I$ where I is some interval like the whole real line or a subset of that. Linear independence of functions is a very strong condition since the all zero solution being the only solution must hold for all t in the interval. Likewise, if the functions are linearly dependent, the same nonzero constants must satisfy the equation $C_1 f_1(t) + \cdots + C_n f_n(t) = 0$ for all t. We can setup a system of equations by differentiating this equation and forming additional equations:

$$\begin{aligned}
&\sum_{k=1}^{n} C_k f_k(t) = 0 \\
&\sum_{k=1}^{n} C_k f_k'(t) = 0 \\
&\sum_{k=1}^{n} C_k f_k''(t) = 0 \\
&\vdots \\
&\sum_{k=1}^{n} C_k f_k^{(n-1)}(t) = 0
\end{aligned}$$

so that we have an $n \times n$ system of equations. We can then write this in matrix form as follows, where the $n \times n$ matrix is known as the Wronskian matrix M_w (whose determinant W is known as the Wronskian determinant):

$$\begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

We can write this for short as $M_w \bar{c} = \bar{0}$ where the matrix $M_w(t)$ is dependent on t. If M_w^{-1} exists then $\bar{c} = M_w^{-1}\bar{0} = \bar{0}$. But since this is a square system, M_w^{-1} exists if $det(M_w(t)) = |W(t)| \neq 0$. So this means that the set of functions will be linearly independent when $|W(t)| \neq 0$. But it turns out this needs to be shown at just a single t on the interval I.

The following lemma holds by virtue of the fact that when $f_1, \ldots f_n$ are linearly dependent, the columns of $M_w(t)$ above will have linear dependency of all $t \in I$.

Lemma 4.1 *If a set of n functions $f_1(t), \ldots, f_n(t)$ defined on the interval $t \in I$ are $(n-1)$ times continuously differentiable and the set is linearly dependent (that is there are constants $C_1, \ldots, C_n$, not all zero, such that $C_1 f_1(t) + \cdots + C_n f_n(t) = 0$), then the Wronskian $W(t)$ (the determinant of the Wronskian matrix) is identically zero, $W(t) = 0$ for all $t \in I$.*

Note that the converse of this statement does not hold without extra conditions! As an example, take the functions $f(t) = t^2$ and $g(t) = t|t|$ on all of $\mathbb{R}$. The Wronskian determinant for these functions is identically zero for all t, but the functions are in fact linearly independent on the whole real line, since there is no constant such that one is a multiple of the other for all t. However, the contrapositive of this statement does hold. Taking the contrapositive of this statement (contrapositive is always true if the statement is true), we have that:

Lemma 4.2 *If a set of n functions $f_1(t), \dots, f_n(t)$ defined on the interval $t \in I$ are $(n-1)$ times continuously differentiable and their Wronskian is nonzero, $W(t) \neq 0$ for some $t \in I$ (need to find just one such $t \in I$), then the set of functions is linearly independent on all of I.*

So to recap, for showing independence, it's enough to show Wronskian is nonzero at one value of t. For showing dependence, Wronskian alone is not of use. Even if we have that $W(t) = 0$ even on the whole interval $t \in I$, we cannot conclude anything about the linear dependence or independence of functions on I without additional conditions. However, if we do know the functions are linearly dependent, then we know the Wronskian identically vanishes on the interval (however, this argument is largely useful only for it's contrapositive).

For example, let:

$$x_1(t) = t \quad \text{and} \quad x_2(t) = t^2$$

Notice that these functions are clearly independent on all of $\mathbb{R}$. Then the Wronskian determinant is:

$$W[x_1(t), x_2(t)] = W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$$

We can conclude that $x_1(t)$ and $x_2(t)$ form a set of linearly independent vector functions on the interval $t \in \mathbb{R}$, just because the Wronskian is nonzero for some $t \in \mathbb{R}$, for example, at $t = 1$. (Note that the Wronskian is zero at zero, but this does not matter as long as it's nonzero at some value of t).

The dimension of a vector space is an integer which is the minimum number vectors which are linerly independent and span the space (hence form a basis for the space). A set of vectors forms a basis for a space if they span the space and are linearly independent. Suppose we have a vector space V and a basis of elements $S = \{u_1, \dots, u_n\}$. Then any element of V can be represented uniquely by a linear combination of elements of S. To show this, suppose for some $v \in V$, we have two representations: $v = C_1 u_1 + \dots C_n u_n$ and $v = D_1 u_1 + \cdots + D_n u_n$. Then it must be that $0 = (C_1 - D_1) u_1 + \cdots + (C_n - D_n) u_n$. But as the n vectors are linearly independent, it follows that $C_k = D_k$ for $k = 1, \dots, n$ and so the two representations are the same. Another very useful result is the following:

Lemma 4.3 *Suppose V is a vector space with $\dim(V) = n$ and $S = \{v_1, \dots, v_n\}$ is a linearly independent set of vectors in V. Then $\operatorname{span}(S) = V$ and so S is a basis for the vector space V.*

To use the above lemma we must know the dimension of the space. For example, $\dim(\mathbb{R}^n) = n$ and $\dim(P_n) = n+1$ (the later is the set of polynomials of degree less than or equal to n). As an example, consider the subspace W of the vector space $V = \mathbb{R}^4$:

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} | x_1 + x_3 = 0, x_2 = x_4 \right\}$$

(To see why this is a subspace of $\mathbb{R}^4$, not just a subset, check that properties $(1),(2)$ of the vector space list called: that is the sum of two elements and any scalar multiple of an element is still in the set). Out of the overdetermined system of equations, we can assign freely two of the variables, for instance $x_3 = \alpha$ and $x_4 = \beta$ with $\alpha, \beta \in \mathbb{R}$. Then we have $x_2 = \beta$ and $x_1 = -\alpha$ and the general set of solutions is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the subspace W is spanned by the vectors $u, v \in \mathbb{R}^4$:

$$u = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Since these two vectors are linearly independent (because they are not multiples of each others) they also form a basis of the set W. The dimension of W is thus 2.

Notice that a set of linearly independent vectors in a vector space V can always be completed into a basis, by adding additional vectors to make it span the space.

Other references:

- Pauls Notes http://tutorial.math.lamar.edu/Classes/DE/Wronskian.aspx
- Wikipedia http://en.wikipedia.org/wiki/Wronskian#The_Wronskian_and_linear_independence
- Basis and Dimension https://www.youtube.com/watch?v=AqXOYgpbMBM
- Various topics https://www.math.ucdavis.edu/~linear/linear.pdf

12/01/2015

Notes on 2×2 systems of ODEs

2×2 systems of ODEs illustrate many concepts found in larger ODE systems and many of their properties generalize directly to the higher order case.

① Nullclines (useful for stability classification of equilibrium points of linear and nonlinear ODE systems).

Suppose the system is
$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

The slope of solution vector $\langle x(t), y(t)\rangle = x(t)\hat{i} + y(t)\hat{j}$ is at any point given by $f(x,y)\hat{i} + g(x,y)\hat{j}$

v-nullcline : $\frac{dx}{dt} = f(x,y) = 0$ and slopes along this nullcline are given by $g(x,y)\hat{j}$ (that is they are vertical and point either up or down depending on the sign of $g(x,y)$).

h-nullcline : $\frac{dy}{dt} = g(x,y) = 0$ and slopes along this nullcline are given by $f(x,y)\hat{i}$ (horizontal, pointing left or right)

Equilibrium point

An (x,y) point where $\frac{dx}{dt} = 0 = \frac{dy}{dt}$

A stable equilibrium attracts (or keeps close) nearby solutions.

An unstable equilibrium repels nearby solutions in at least one direction.

Ex) $\frac{dx}{dt} = 1-x-y = f(x,y)$

$\frac{dy}{dt} = 1-x^2-y^2 = g(x,y)$

h-nullcline: $\langle \frac{dx}{dt}, 0 \rangle \Rightarrow g(x,y) = 0 = 1-x^2-y^2$

$\Rightarrow x^2+y^2=1$ (which is a circle of radius 1).

v-nullcline: $\langle 0, \frac{dy}{dt} \rangle \Rightarrow f(x,y) = 0 = 1-x-y$

$\Rightarrow x+y=1$ (line)

equilibrium points: $f(x,y) = 0 = g(x,y)$

From $x+y=1 \Rightarrow x=(1-y)$

$\Rightarrow x^2+y^2=1 = \cancel{x}(1-y)^2+y^2 = y^2-2y+1+y^2=1$

$\Rightarrow 2y^2-2y=0 \Rightarrow y^2-y=0 \Rightarrow y(y-1)=0$

$\Rightarrow y=0$ or $y=1$

For $y=0$, $x=1-0=1 \Rightarrow (1,0)$ is eq point

For $y=1$, $x=1-1=0 \Rightarrow (0,1)$ is an eq point.

So the two eq points are $(1,0)$ and $(0,1)$. To analyze their stability, we plot the slopes along the nullclines.

slopes along h-nullcline: $\frac{dx}{dt}\hat{i} = f(x,y)\hat{i} = (1-x-y)\hat{i}$

$(1-x-y)>0$ when $y<1-x$ and

$(1-x-y)<0$ when $y>1-x$ and notice that

the line $y=1-x$ represents the v-nullcline.

slopes along v-nullcline: $\frac{dy}{dt}\hat{j} = g(x,y)\hat{j} = (1-x^2-y^2)\hat{j}$

when $1-x^2-y^2>0 \Rightarrow y^2<1-x^2$ (slopes point up)

when $1-x^2-y^2<0 \Rightarrow y^2>1-x^2$ (slopes point down)

The following picture results:

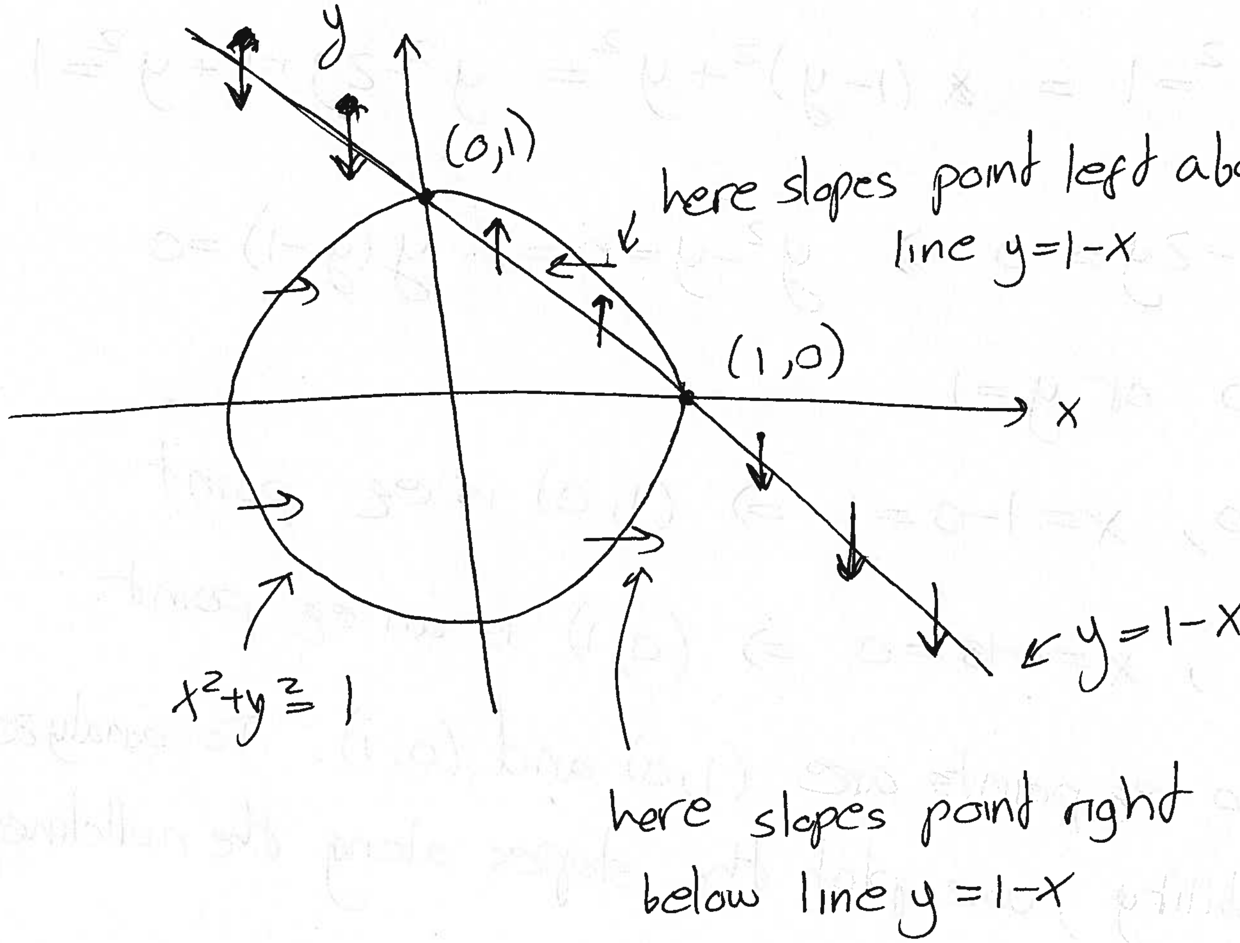
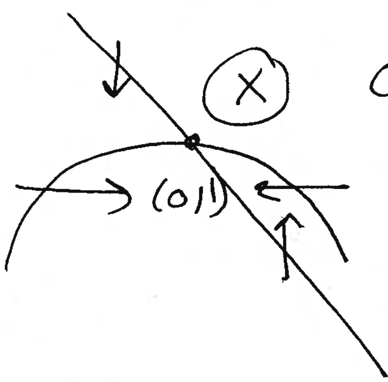


Fig 1

along v-nullcline (line $y=1-x$) slopes point up inside circle (when $y^2 < 1-x^2$) and point down outside the circle (when $y^2 > 1-x^2$).

We can definitively say (1,0) is <u>unstable</u> eq point because we see that solution slopes point away (in opposite directions) around the point, at least along the line $y=1-x$ (and we only need to demonstrate one direction along which nearby solutions go in opposite directions)

From Fig 1 it appears that $(0,1)$ is a <u>stable</u> equilibrium point and this turns out to be the case. However, one should check the remaining nearby direction to be sure of this.



check here for instance

since the slope of solution at any (x,y) is given by $f(x,y)\hat{i} + g(x,y)\hat{j} = (1-x-y)\hat{i} + (1-x^2-y^2)\hat{j}$

we can plug in for instance $x=0.5$, $y=1$.

$\Rightarrow$ slope at $(0.5,1)$ is $(1-0.5-1)\hat{i} + (1-0.5^2-1^2)\hat{j}$

$= -0.5\hat{i} - 0.5^2\hat{j}$ which points towards $(0,1)$.

Thus, we mark $(0,1)$ as a stable eq point since it appears that along all directions, solutions are attracted towards it.

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Thus, we mark (0,1) as a stable eq point since it appears that along all directions, solutions are attracted towards it.

Systems of ODE via nullclines: (2.6)

Ex] $\frac{dx}{dt} = x(1-x-y)$; $\frac{dy}{dt} = 2y\left(1-\frac{y}{2}-\frac{3}{2}x\right)$

solution vector field is $\langle x(t), y(t) \rangle$

solution slope vector field is $\langle x'(t), y'(t) \rangle$

vertical nullclines: $\langle 0, y'(t) \rangle$ (when $x'(t)=0$)

horizontal nullclines: $\langle x'(t), 0 \rangle$ (when $y'(t)=0$)

vn: $x'(t) = x(1-x-y) = 0$

$\Rightarrow \underline{x=0}$ or $1-x-y=0 \Rightarrow x+y=1 \Rightarrow \underline{y=1-x}$

hn: $y'(t) = 2y\left(1-\frac{y}{2}-\frac{3}{2}x\right) = 0$

$\Rightarrow \underline{y=0}$, $1-\frac{y}{2}-\frac{3}{2}x=0 \Rightarrow \underline{y=2-3x}$

Equilibrium points satisfy $x'(t)=0=y'(t)$

$x(1-x-y) = 0 = 2y\left(1-\frac{y}{2}-\frac{3}{2}x\right)$

lhs zero when $x=0$

$\Rightarrow$ rhs $= 2y\left(1-\frac{y}{2}\right) = 0$ when $y=0$, $y=2$

$\Rightarrow$ points $(0,0)$ and $(0,2)$

lhs zero when $x+y=1 \Rightarrow y=1-x$

$$\Rightarrow \text{rhs} = 2(1-x)\left[1-\frac{1}{2}(1-x)-\frac{3}{2}x\right]$$

$$= 2(1-x)\left[1-\frac{1}{2}+\frac{1}{2}x-\frac{3}{2}x\right]$$

$$= 2(1-x)\left(\frac{1}{2}-x\right) = (1-x)(1-2x)$$

$\Rightarrow x=1$, $x=\frac{1}{2}$

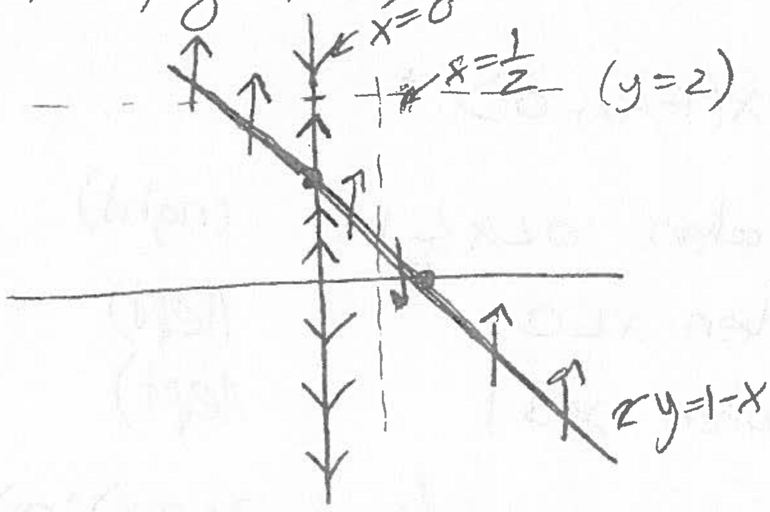
$\Rightarrow$ points $(1,1-1)=(1,0)$; $\left(\frac{1}{2},1-\frac{1}{2}\right)=\left(\frac{1}{2},\frac{1}{2}\right)$

So 4 equilibrium points: $(0,0),(0,2),(1,0),\left(\frac{1}{2},\frac{1}{2}\right)$

Next, we must characterize these as stable or unstable. For this we must draw the nullclines. Here its easiest to draw the two sets separately.

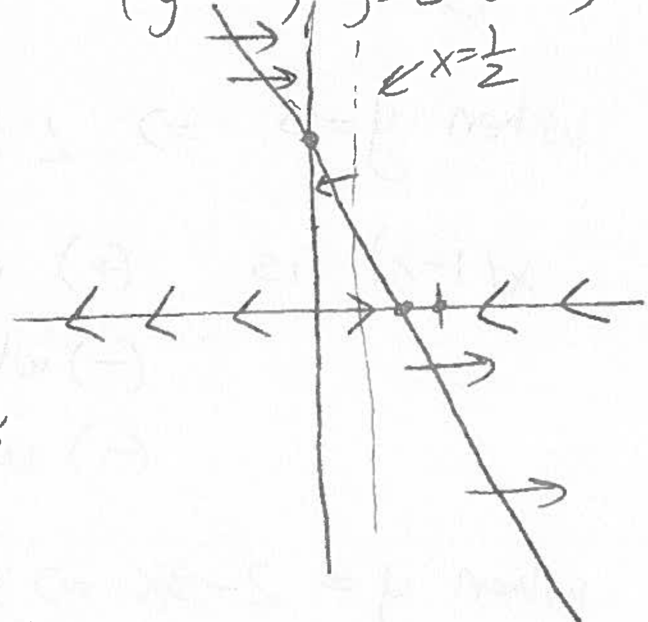
vertical nullclines:

$(x=0,\ y=1-x)$



horizontal nullclines:

$(y=0,\ y=2-3x)$



along vn slopes are $\langle 0, 2y(1-\frac{y}{2}-\frac{3}{2}x)\rangle$

when $x=0 \Rightarrow$ slopes: $\langle 0, 2y(1-\frac{y}{2})\rangle$

$2y(1-\frac{y}{2}) = y(2-y)$ is (−) when $y<0$ (down

(+) when $0<y<2$ (up

(−) if $y>2$ (down

when $y=1-x \Rightarrow$ slopes $\langle 0, 2(1-x)[1-\frac{1}{2}(1-x)-\frac{3}{2}x]\rangle$

$= \langle 0, (1-x)(1-2x)\rangle$

$(1-x)(1-2x) = (x-1)(2x-1)$ is (+) for $x>1$ (up)

(−) for $\frac{1}{2}<x<1$ (dow

(+) for $x<\frac{1}{2}$ (up)

along hn: slopes are $\langle x(1-x-y), 0 \rangle$

when $y=0 \Rightarrow \langle x(1-x), 0 \rangle$

$x(1-x)$ is (+) when $0<x<1$ (right)

(−) when $x<0$ (left)

(−) when $x>1$ (left)

when $y = 2-3x \Rightarrow$ slope: $\langle x(1-x-2+3x); 0 \rangle =$

$= \langle x(-1+2x), 0 \rangle = \langle x(2x-1), 0 \rangle$

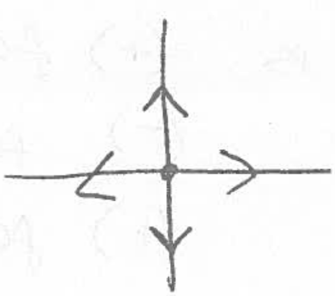
$x(2x-1)$ is (+) when $x<0$ (right)

(−) when $0<x<\frac{1}{2}$ (left)

(+) when $x>\frac{1}{2}$ (right)

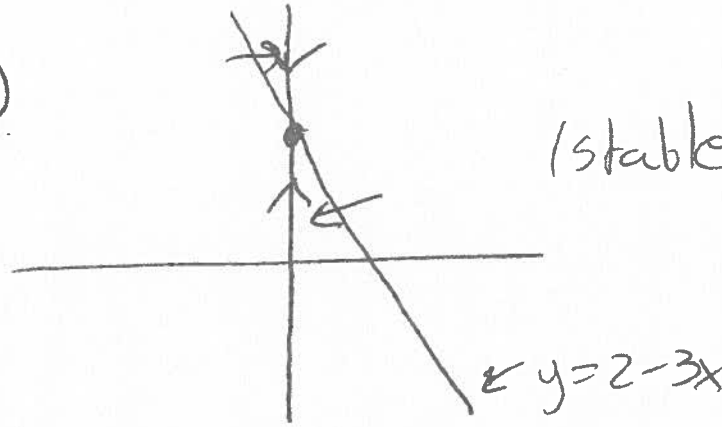
To classify the eq points, we must look at the slope directions around each point from the nullcline plots.

(A) $(0,0)$



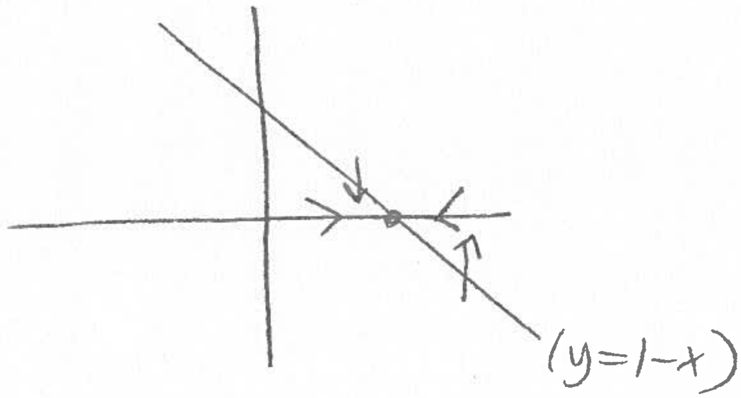
(unstable)

(B) $(0,2)$



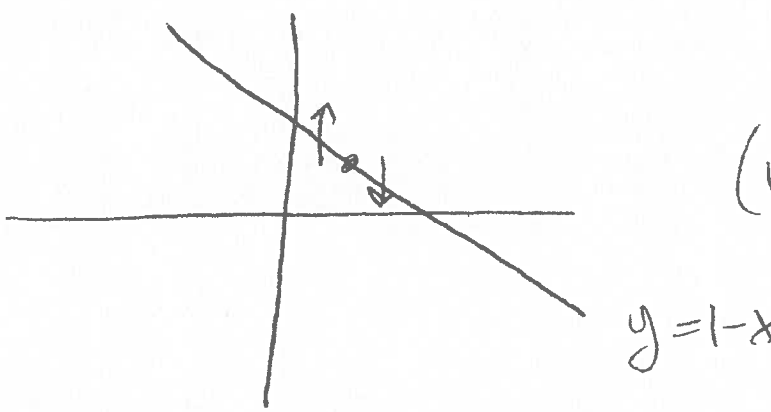
(stable)

(C) $(1,0)$



(stable)

(D) $\left(\frac{1}{2}, \frac{1}{2}\right)$



(unstable)

Systems of Linear Equations: Row Reduction

Given a system of equations, we rewrite in matrix form $Ax = b$, form the augmented matrix $M = [A|b]$ with b as last column vector of A and row reduce: $E_p E_{p-1} \cdots E_1 M = \tilde{M}$.

We apply enough elementary row operations so that $\tilde{M} = \mathrm{rref}(M)$ (row reduced echelon form). In some cases, it is enough to reduce to row echelon form.

A matrix is in row reduced echelon form if:

(1) The first non-zero element in each row (the pivot element) is a 1.

(2) Each leading entry (pivot) is in a column to the right of the leading entry in the previous row.

(3) Rows with all zero elements are at the bottom.

(4) The leading entry in each row is the only non-zero entry in its column.

Note: If (1)-(3) satisfied, but not (4): matrix is in row echelon form.

matrices in rref (reduced row echelon form):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

matrices in row echelon form (need few more elementary operations to get into rref form):

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Ex) what form is $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in?

Answer: It is in <u>rref</u> form.

Elementary transformations:

(1) swap rows

(2) multiply a row by a constant

(3) add a multiple of another row to a row.

All these can be accomplished by multiplication (from left) with elementary matrices.

Here a 3 examples of 3x3 systems with unique, no sol, inf. many sols.

Ex)

$$\left.\begin{aligned} x+2y-z&=3 \\ x+3y+z&=5 \\ 3x+8y+4z&=17 \end{aligned}\right\} \quad M_1 = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & 8 & 4 & 17 \end{bmatrix}$$

$$M_1 \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 4/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 11/3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 4/3 \end{bmatrix}$$

$$\sim \left[\begin{array}{cccc} 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -2/3 \\ \underbrace{0 \quad 0 \quad 1}_{\text{Identity matrix}} & & & 4/3 \end{array}\right] \iff \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17/3 \\ -2/3 \\ 4/3 \end{pmatrix}$$

$\Rightarrow$ unique solution: $x = \frac{17}{3}$; $y = -\frac{2}{3}$; $z = \frac{4}{3}$

Ex)

$$\left.\begin{aligned} x-2y+4z&=2 \\ 2x-3y+5z&=3 \\ 3x-4y+6z&=7 \end{aligned}\right\} \quad M_2 = \begin{bmatrix} 1 & -2 & 4 & 2 \\ 2 & -3 & 5 & 3 \\ 3 & -4 & 6 & 7 \end{bmatrix}$$

This is an inconsistent set of equations.

$$M_2 = \begin{bmatrix} 1 & -2 & 4 & 2 \\ 2 & -3 & 5 & 3 \\ 3 & -4 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & -6 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last row of the matrix in row echelon form (we do not even need the rref here) says:

$0 \cdot x + 0 \cdot y + 0 \cdot z = 1$ which is impossible.

So the set of equations is inconsistent and has <u>no solution</u>.

Ex

$$\left.\begin{aligned} x + y + 3z &= 1 \\ 2x + y - z &= 3 \\ 5x + 7y + z &= 7 \end{aligned}\right\} \quad M_3 = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{bmatrix}$$

$$M_3 \sim \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & -7 & 1 \\ 0 & 2 & -14 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third row all zeros simply means that one of the equations is a multiple of another equation or a sum of multiples of other equations (i.e. one of the equations carries no new information). It is linearly dependent with the others.

There are infinitely many solutions.

$$\operatorname{rref}(M_3) = \begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} x + 10z = 0 \\ y - 7z = 1 \end{array} \Rightarrow \begin{array}{ll} x = -10z & = -10\alpha \\ y = 7z + 1 & = 7\alpha + 1 \\ z = \alpha & \end{array}$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10\alpha \\ 7\alpha + 1 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -10 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

for any $\alpha \in \mathbb{R}$ is a solution.

Null space example

Ex) Find Null(A) for

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\text{Null}(A) = \{ \vec{x} \in \mathbb{R}^5 \mid A\vec{x} = \vec{0} \}$$

We find all solutions to $A\vec{x} = \vec{0}$ by row reducing $M = [A \mid \vec{0}]$. Since rhs is $\vec{0}$, we can simply row reduce A.

$$A \to \begin{pmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ +1 & +1 & -2 & +3 & -1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -3/2 & 3 & -3/2 \\ 0 & 0 & -3/2 & 0 & -3/2 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$ swa row her

$$\to \begin{pmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$ replace last row bi last row +3rd row to make it all zer

$$\rightarrow \begin{pmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \varepsilon\ R_2^* = R_2 - R_3$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = rref(A)$$

row reduced echelon form of A.

From this, we conclude:

$$\left.\begin{aligned} x_1 + x_2 + x_5 &= 0 \\ x_3 + x_5 &= 0 \\ x_4 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= -x_2 - x_5 = -t - s \\ x_2 &= t \\ x_3 &= -x_5 = -s \\ x_4 &= 0 \\ x_5 &= s \end{aligned}$$

$$\Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -t-s \\ t \\ -s \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So $Null(A) = \left\{ s\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for } s,t \in \mathbb{R} \right\}$

Notes on Linear Dependence / Independance / Basis

A finite non-empty <u>set</u> of vectors $\{\vec{v}_1, \dots, \vec{v}_r\}$ with each $\vec{v}_j \in \mathbb{R}^n$ is:

(a) <u>linearly dependent</u>, if there exist scalars $k_1, \dots, k_r$ not all zero, such that:

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_r \vec{v}_r = \vec{0}$$

This means some of the vectors can be written as linear combination of the others.

E.g. suppose $k_j \neq 0$. Then:

$$k_j \vec{v}_j = \sum_{i=1, i \neq j}^{r} k_i \vec{v}_i \Rightarrow \vec{v}_j = \frac{1}{k_j} \sum_{i=1, i \neq j}^{r} k_i \vec{v}_i$$

(b) <u>linearly independent</u>, if the only values $k_1, \dots, k_r$ for which $k_1 \vec{v}_1 + \dots + k_r \vec{v}_r = \vec{0}$ are all zero (i.e. $k_1 = \dots = k_r = 0$).

This means none of the vectors can be written as a linear combination of the others.

To test a set of vectors for linear independence, we use determinant or row reduction depending on n and r.

(1) $\underline{n=r}$

We have a square system here.

$$\underbrace{[\vec{v}_1, \ldots, \vec{v}_r]}_{M} \begin{bmatrix} k_1 \\ \vdots \\ k_r \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

If $\det(M) \neq 0 \Rightarrow M^{-1}$ exists $\Rightarrow \text{null}(M) = \{0\}$.
$\Rightarrow$ set is linearly independent.

If $\det(M) = 0 \Rightarrow M^{-1}$ does not exist
and $\text{null}(M)$ contains other nonzero vectors besides $\vec{0}$.
$\Rightarrow$ set is linearly dependent.

(2) $\underline{r > n}$

The set $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly dependent in this case. If we take $M = [\vec{v}_1, \ldots, \vec{v}_r]$ then $\text{rank}(M) \leq n$. This means at most n of the vectors are linearly independent. To find out, we must row reduce and find the pivot columns.

(3) $\underline{r < n}$

The set $\{\vec{v}_1, \ldots, \vec{v}_r\}$ may be linearly dependent or independent. Must check that $\text{rank}(M) = r$ to conclude linear independence.

Ex Is the set of these polynomials linearly dependent or independent?

$$p_1 = 1 - x \; ; \quad p_2 = 5 + 3x - 2x^2 \; ; \quad p_3 = 1 + 3x - x^2$$

Note that $3p_1 - p_2 + 2p_3 = 0$ so they are linearly dependent. To see this, let

$$p(x) = \alpha x^2 + \beta x + \gamma \iff \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$p_1 \iff \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \; ; \quad p_2 \iff \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} \; ; \quad p_3 \iff \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & -2 & -1 \\ -1 & 3 & 3 \\ 1 & 5 & 1 \end{pmatrix} \qquad r = n \text{ so check } \det(M)$$

$$\det(M) = (-1)(-1)^{2+1} \begin{vmatrix} -2 & -1 \\ 5 & 1 \end{vmatrix} + (1)(-1)^{3+1} \begin{vmatrix} -2 & -1 \\ 3 & 3 \end{vmatrix}$$

$$= (-2+5) + (-6+3) = 3 - 3 = 0$$

Since $\det(M) = 0$, set of polynomials is linearly dependent.

Given $A \in \mathbb{R}^{m \times n}$, we can write A in terms of column vectors or row vectors.

$$A = [\vec{a}_1, \dots, \vec{a}_n] = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}$$

rank(A) = # of linearly indep rows = # of linearly indep columns $\leq \min(m,n)$.

$\mathrm{Col}(A)$ = column space of A = range(A)
$= \mathrm{span}\{\vec{a}_1, \dots, \vec{a}_n\}$

$\mathrm{Row}(A)$ = row space of A = $\mathrm{span}\{\vec{r}_1, \dots, \vec{r}_m\}$

$\mathrm{Col}(A) \neq \mathrm{Col}(\mathrm{rref}(A))$ but $\mathrm{Row}(A) = \mathrm{Row}(\mathrm{rref}(A))$

$\boxed{\text{Ex}}$ Let $A = \begin{pmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix}$

$$A \to \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 3 & 8 & -3 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & -\frac{9}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{ref}(A)$$

row echelon form

rank(A) = 2 (2 pivots)

$$\to \begin{pmatrix} 1 & 0 & \frac{17}{7} & -\frac{17}{7} \\ 0 & 1 & -\frac{9}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A)$$

(row <u>reduced</u> echelon form
pivots only nonzeros in column

$$\text{Row}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 17/7 \\ -17/7 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 1 \\ -9/7 \\ 2/7 \end{pmatrix}^T \right\}$$ these are rows of rref(A)

(T = transpose since these are row vectors)

$$\text{Col}(A) = \text{span}\Bigg\{ \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix}}_{\text{these are columns of } A \text{ corresponding to pivots of rref}(A)} \Bigg\}$$

$$\text{Null}(A) = \{ \vec{x} \mid A\vec{x} = \vec{0} \}$$

Since $\text{rref}(A) = \begin{pmatrix} 1 & 0 & \frac{17}{7} & -\frac{17}{7} \\ 0 & 1 & -\frac{9}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 17/7 & -17/7 \\ 0 & 1 & -9/7 & 2/7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + \tfrac{17}{7}x_3 - \tfrac{17}{7}x_4 &= 0 \\ x_2 - \tfrac{9}{7}x_3 + \tfrac{2}{7}x_4 &= 0 \end{aligned}\right\} \text{ let } x_3 = s \text{ and } x_4 = t$$

Then $x_1 = -\frac{17}{7}x_3 + \frac{17}{7}x_4 = -\frac{17}{7}s + \frac{17}{7}t$ $\quad x_3 = s$

$x_2 = \frac{9}{7}s - \frac{2}{7}t$ $\quad x_4 = t$

$$\Rightarrow \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} -17/7 \\ 9/7 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 17/7 \\ -2/7 \\ 0 \\ 1 \end{pmatrix}, \quad s,t \in \mathbb{R}$$

<u>basis</u> for null(A) $= \left\{ \begin{pmatrix} -17/7 \\ 9/7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 17/7 \\ -2/7 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\text{null}(A) = \text{span}\left\{ \begin{pmatrix} -17/7 \\ 9/7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 17/7 \\ -2/7 \\ 0 \\ 1 \end{pmatrix} \right\}$$

If $\underline{V}$ is any vector space and $S = \{\vec{v}_1, \dots, \vec{v}_r\}$ is a set of vectors in V, then S is a <u>basis</u> for $\underline{V}$ if:

(a) S is linearly independent

(b) S spans $\underline{V}$.

Linear Independence of functions and vector functions

A set of functions $\{f_1(t) \dots, f_n(t)\}$ is linearly dependent ~~when~~ on interval I when the equation

$$K_1 f_1(t) + \dots + K_n f_n(t) = 0$$

has nonzero solution (i.e. not all $K_1, \dots, K_n$ zero) for all $t \in I$.

If $\{f_1(t), \dots, f_n(t)\}$ are $(n-1)$ times continuously differentiable on I, we can write:

$$K_1 f_1(t) + \dots + K_n f_n(t) = 0$$

$$K_1 f_1'(t) + \dots + K_n f_n'(t) = 0$$

$$K_1 f_1^{(2)}(t) + \dots + K_n f_n^{(2)}(t) = 0$$

$$\dots$$

$$K_1 f_1^{(n-1)}(t) + \dots + K_n f_n^{(n-1)}(t) = 0$$

$$\Rightarrow \begin{pmatrix} f_1(t) & \cdots & f_n(t) \\ \vdots & & \vdots \\ f_1^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{pmatrix} \begin{pmatrix} K_1 \\ \vdots \\ K_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow M_W(t) \begin{pmatrix} K_1 \\ \vdots \\ K_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thm^A when $\{f_1(t), \dots, f_n(t)\}$ are linearly dependent on I, then $\det(M_W(t)) = W(t) = 0$ for all $t \in I$.

Thm^B (contrapositive of previous thm^A). If it so happens that $\det(M_W(t^*)) = W(t^*) \neq 0$ for some $t^* \in I$ (at least for one value of t), then we can conclude that $\{f_1(t) \dots, f_n(t)\}$ are linearly independent on <u>all of I</u>.

$\boxed{\text{Ex}}$ show $f(t) = t$ and $g(t) = t^2$ are linearly indep. on all of $\mathbb{R}$. Both defined and differentiable on $I = \mathbb{R}$.

$$M_w(t) = \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}$$

$$W(t) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} \ ; \ W(1) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 \neq 0$$

Since $W(1) \neq 0$, $f(t)$ and $g(t)$ linearly indep on $I = \mathbb{R}$.

Notice that the converse of Thm A may not hold. That is, even if $\det(M_w(t)) = 0$ for all $t \in I$, the functions may be linearly independent.

Ex) Does the set $S = \{t+1, t^2+1, t^2-t\}$ span $\mathbb{P}_2$ (vector space of polynomials of degree ≤ 2)

If it did then any $p \in \mathbb{P}_2$ can be written as a linear combination of polynomials in S.

Let $p(t) = at^2 + bt + c$

$$\Rightarrow at^2 + bt + c = k_1(t+1) + k_2(t^2+1) + k_3(t^2 - t)$$

Is there a unique solution $\{k_1, k_2, k_3\}$ for each $\{a, b, c\}$?

$$at^2+bt+c = t^2[k_2+k_3] + t[k_1-k_3] + [k_1+k_2]$$

$$\Rightarrow \quad \begin{aligned} k_2+k_3 &= a \\ k_1-k_3 &= b \\ k_1+k_2 &= c \end{aligned}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}}_{M} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

note that $\det(M) = (1)(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + (1)(-1)^{1+3}\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}$

$$= (-1)(0-1) + (1)(-1-0) = 1-1=0$$

Hence for every $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ there is no unique solution $\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$. Let's see what subset of $\mathbb{P}_2$ these vectors span

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 0 & -1 & b \\ 1 & 1 & 0 & c \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & c \\ 1 & 0 & -1 & b \\ 0 & 1 & 1 & a \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & c \\ 1 & 1 & 0 & a+b \\ 0 & 1 & 1 & a \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & c \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & a+b-c \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & c-a \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & a+b-c \end{array}\right)$$

Consisdend only when $a+b-c=0$

$$\Rightarrow c = a+b$$

Thus, the set S spans only a subset of polynomials in $\mathbb{P}_2$ of the form $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$

i.e. $p(t) = at^2 + bt + c$ is an element of this subset if $c = a+b$.

Basis and dimension

Thm Every set of linearly independent vectors which spans a finite dimensional vector space $\underline{V}$ is a basis for $\underline{V}$.

Thm Every finite dimensional vector space $\underline{V}$ has at least one basis. (But there can be many bases!)

Ex $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$ and so is $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$

Thm Any two bases of a finite dimensional vector space $\underline{V}$ contain the same number of elements.

This unique number of vectors in a basis of $\underline{V}$ is called the dimension of $\underline{V}$.

Thm Let $\underline{V}$ denote a vector space and

$S = \{\vec{u}_1, \ldots, \vec{u}_n\}$ be a basis for $\underline{V}$. Then:

(a) S is linearly independent

(b) $\text{span}(S) = \underline{V}$

(c) Every element of $\underline{V}$ can be written in a unique way as a linear combination of vectors in S.

Proof of (c) Suppose $\vec{v} \in \underline{V}$ and we have two representations:

$$\vec{v} = c_1\vec{u}_1 + \ldots + c_n\vec{u}_n \quad \text{and} \quad \vec{v} = d_1\vec{u}_1 + \ldots + d_n\vec{u}_n$$

Then we must have $c_j = d_j$ for $j = 1, \ldots, n$. This is because subtracting we get:

$$\vec{0} = (c_1 - d_1)\vec{u}_1 + \ldots + (c_n - d_n)\vec{u}_n$$

but by linear independence, we must have

$$c_1 - d_1 = 0, \ldots, c_n - d_n = 0 \Rightarrow c_j = d_j \text{ for } j = 1, \ldots, n$$

Again recall that $\dim(\underline{V}) = \#$ of elements in basis of $\underline{V}$.

Ex $\dim(\mathbb{R}^n) = n$ and $\dim(\mathbb{P}_n) = n+1$

set of polynomials of degree $\leq n$.

$\boxed{\text{If } \dim(\underline{V}) = n, \text{ any set of } n \text{ linearly indep. vectors spans } \underline{V}}$

$\boxed{\text{Ex}}$ Does the set of vectors

$$\{(1,1,1), (3,2,1), (1,1,0), (1,0,0)\} \text{ span } \mathbb{R}^3?$$

Notice that $\dim(\mathbb{R}^3) = 3$. In order to span $\mathbb{R}^3$ we need 3 linearly independent vectors. We can stack the vectors as rows or columns of a matrix and check rank.

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(M) = 3$ so there are 3 linearly independent vectors in the set and they span $\mathbb{R}^3$. (since $\dim(\mathbb{R}^3) = 3$).

Ex) The elements $\{1, t, t^2\}$ span $\mathbb{P}_2$.

$\dim(\mathbb{P}_2) = 3$. The elements correspond to the vectors

$$p(t) = at^2 + bt + c \Leftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow 1 \Leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; t \Leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; t^2 \Leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ and these three are linearly independent.