Second Order Equations with Constant Coefficients

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Textbook: Section 4.1 - 4.3

The general homogeneous second order ODE with constant coefficients is:

$$L(y) = ay''(t) + by'(t) + cy(t) = 0 \tag{0.1}$$

where $a, b, c \in \mathbb{R}$ (and $a \neq 0$). If we plug in $y(t) = e^{rt}$ into this equation we get:

$$e^{rt}\left(ar^2 + br + c\right) = 0 \implies ar^2 + br + c = 0$$

This means that $y(t) = e^{rt}$ is a solution of (0.1) as long as $ar^2+br+c = 0$. The characteristic equation $ar^2 + br + c = 0$ has solutions:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{d}}{2a}$$

We analyze three different cases in which the roots of the equation fall. In each case, the general solution to (0.1) is a linear combination of two linearly independent solutions. This must be the case, because the vector space of solutions to (0.1) has dimension two.

- (A) The equation $ar^2+br+c = 0$ has two distinct roots r_1, r_2. This occurs when the discriminant $d = b^2 - 4ac > 0$. This means that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are both linearly independent solutions, since $r_1 \neq r_2$. Then the general solution to (0.1) is:

  $$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

  An example is $y''+5y'+6y = 0$. Then the characteristic equation is $r^2+5r+6 = (r+2)(r+3) = 0$ and has roots $r_1 = -2, r_2 = -3$. Thus, the general solution is $y(t) = C_1 e^{-2t} + C_2 e^{-3t}$.

- (B) The equation $ar^2 + br + c = 0$ has a repeated root when $b^2 - 4ac = 0$. This root $r = -\frac{b}{2a}$. One solution is given by $y_1(t) = e^{rt}$. For the second solution, we use the variation of parameters idea. We set $y_2(t) = v(t)e^{rt}$. We like to find a function $v(t)$ such that $L(y_2(t)) = 0$. We get:

  $$\begin{aligned} y_2'(t) &= v'e^{rt} + rve^{rt} \\ y_2''(t) &= v''e^{rt} + rv'e^{rt} + r^2ve^{rt} + rv'e^{rt} = v''e^{rt} + 2rv'e^{rt} + r^2ve^{rt} \end{aligned}$$

Plugging this into $ay_2''(t) + by_2'(t) + cy_2(t) = 0$, we get:

$$\begin{aligned} av''e^{rt} &+ 2arv'e^{rt} + ar^2ve^{rt} + bv'e^{rt} + brve^{rt} + cve^{rt} \\ &= av''e^{rt} + v'e^{rt}\left[2ar + b\right] + ve^{rt}\left[ar^2 + br + c\right] = 0 \end{aligned}$$

Notice that $ar^2 + br + c = 0$ and since $r = -\frac{b}{2a}$, we get $2ar + b = -b + b = 0$. Hence, we have that:

$$av''e^{rt} = 0 \implies v''(t) = 0 \implies v(t) = t + C$$

For simplicity, we can take $C = 0$, to get $v(t) = t$. Hence, the general solution to (0.1) in the case of repeated root is:

$$y(t) = C_1e^{rt} + C_2te^{rt}$$

As an example, let $y'' - 4y' + 4y = 0$. Then $r^2 - 4r + 4 = (r - 2)^2 = 0 \implies r_1 = r_2 = r = 2$. Thus, the general solution is $y(t) = C_1e^{2t} + C_2te^{2t}$. Notice that if initial conditions such as $y(0) = 1$, $y'(0) = -1$ are given, we must use the product rule when differentiating $y(t)$.

- (C) The equation $ar^2 + br + c = 0$ has complex conjugate roots when $b^2 - 4ac < 0$. If we set $d = b^2 - 4ac$ the roots are given by:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{d}}{2a} = \frac{-b \pm \sqrt{-1}\sqrt{-d}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{-d}}{2a} = \alpha \pm i\beta$$

Then the general solution to (0.1) is:

$$y(t) = C_1e^{\alpha t}\cos(\beta t) + C_2e^{\alpha t}\sin(\beta t) = e^{\alpha t}\left(C_1\cos(\beta t) + C_2\sin(\beta t)\right)$$

We will make use of Euler's identity:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

The complex valued solution using $r = \alpha + i\beta$ is:

$$y_{\text{comp}}(t) = e^{rt} = e^{\alpha t + i\beta t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t)\right]$$

Notice that since $L[y_{\text{comp}}(t)] = 0$, it follows that if we extract the real and imaginary parts of $y_{\text{comp}}(t)$ then $L[Re[y_{\text{comp}}(t)]] = 0 = L[Im[y_{\text{comp}}(t)]]$ (that is, both $Re[y_{\text{comp}}(t)] = e^{\alpha t}\cos(\beta t)$ and $Im[y_{\text{comp}}(t)] = e^{\alpha t}\sin(\beta t)$ are real valued linearly indepedent solutions to (0.1)). Hence, the general solution in the case of complex conjugate roots is:

$$y(t) = C_1e^{\alpha t}\cos(\beta t) + C_2e^{\alpha t}\sin(\beta t)$$

with $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{-d}}{2a}$. Notice that when $\alpha < 0$ in this case, $\lim_{t\to\infty} y(t) = 0$. This happens when both a and b have the same sign.

As an example, consider $y'' + 2y' + 4y = 0$. Then the corresponding characteristic equation is $r^2 + 2r + 4 = 0$. Then $r_1, r_2 = \frac{-2\pm\sqrt{4-16}}{2} = -1 \pm i\frac{\sqrt{12}}{2} = -1 \pm i\sqrt{3}$ since $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$. The general solution is thus given by:

$$y(t) = C_1e^{-t}\cos(\sqrt{3}t) + C_2e^{-t}\sin(\sqrt{3}t)$$

For more examples, see: http://tutorial.math.lamar.edu/Classes/DE/IntroSecondOrder.aspx.

1 Method of Undetermined Coefficients

Textbook: Section 4.4

The solution to the non-homogeneous equation

$$L(y) = ay''(t) + by'(t) + cy(t) = f(t) \tag{1.1}$$

for some nonzero $f(t)$ is given as a sum of the solution y_h to the corresponding homogeneous equation $L(y_h) = 0$ (given by $ay''(t) + by'(t) + cy(t) = 0$) and a particular solution y_p which solves the non-homogenenous $L(y_p) = f(t)$ (i.e. the general solution $y(t) = y_h(t) + y_p(t)$). To find the homogeneous equation solution, we simply refer to the above three cases. For the particular solution, we can use either the method of undetermined coefficients of the method of variation of parameters (the later method was not covered during this term).

For the method of undetermined coefficients, we can choose a trial function $y_p(t)$ involving some undetermined constants based on $f(t)$. If for example $f(t) = \sin(3t)$ we will use $y_p(t) = A\cos(3t) + B\sin(3t)$ for the particular solution guess. We then plug in into $L(y_p) = f(t)$ to find the constants A and B which work. Note that for trigonometric functions we must include both cos and sin in the particular solution. If $f(t)$ is a product of two functions, for example $f(t) = t\cos(2t)$ then we would use as our particular solution candidate, $y_p(t) = (A + Bt)(C\cos(2t) + D\sin(2t))$ since t corresponds to a first degree polynomial.

However, care must be taken if the solution to the homogeneous equation $y_h(t)$ is found to contain terms which are linearly dependent with terms in the proposed particular solution $y_p(t)$. In that case, we must multiply the proposed solution $y_p(t)$ by factor t to some power. Notice that if we are dealing with an equation in terms of independent variable x then we replace the t in this discussion by x. The power is the minimum power of t such that the new resulting particular solution has no linearly dependent terms with terms of $y_h(t)$. For second order linear constant coefficient equations, it is always the case that $y_h(t)$ is a linear combination of two linearly indepedent solutions. For example, consider the equation $y'' - 2y' + y = 3e^t$. The corresponding homogeneous equation is $y_h'' - 2y_h' + y_h = 0$. The corresponding characteristic equation is $r^2 - 2r + 1 = 0 = (r-1)^2$ yielding the root of multiplicity 2, $r = 1$. So that $y_h(t) = C_1e^t + C_2te^t$. Based on $f(t) = 3e^t$ we set $\tilde{y}_p(t) = Ae^t$ but this corresponds to the term C_1e^t in the homogeneous solution and will not work (try plugging this in and see). Then, we try instead $\tilde{y}_p(t) = t(Ae^t) = Ate^t$ but this is linearly dependent with C_2te^t in the homogeneous solution. So we must take $y_p(t) = t^2(Ae^t) = At^2e^t$ for the particular solution and plug into $L(y_p) = f(t)$. We get:

$$y_p'(t) = 2Ate^t + At^2e^t \quad \text{and} \quad y_p''(t) = 2Ae^t + 2Ate^t + 2Ate^t + At^2e^t$$

Upon plugging into $y_p'' - 2y_p' + y_p = 3e^t$, we get:

$$(2Ae^t + 4Ate^t + At^2e^t) - 2(2Ate^t + At^2e^t) + At^2e^t = 3e^t \implies 2Ae^t = 3e^t$$

Thus, $A = \frac{3}{2}$ so that $y_p(t) = \frac{3}{2}t^2e^t$ and $y(t) = y_h(t) + y_p(t) = C_1e^t + C_2te^t + \frac{3}{2}t^2e^t$. See attachment for more examples.

References with examples:
http://tutorial.math.lamar.edu/Classes/DE/UndeterminedCoefficients.aspx,

2 Simple Harmonic Oscillator

Textbook: Section 4.2 - 4.6

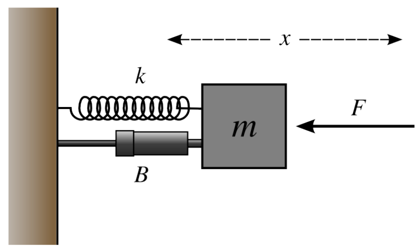


Figure 1: SHM setup: spring constant k, damping coefficient b and possible external force F. Motion is one dimensional along x.

SHM offers a very good physical example for second order differential equations. The net force on the block is equal to its mass times its acceleration, giving:

$$mx'' = -kx - bx' + F(t)$$

Notice that both the spring and damping oppose the motion of the block, but the spring force (by Hooke's law) is dependent on the displacement of the block (x) while the damping force is proportional to the velocity of the block (x', think of running in water). The initial conditions are $x(0) = x_0$ (displacement to the left or right of equilibrium position $x = 0$) and $x'(0) = v_0$ (push to left or right).

Notice that when $F = 0$ (unforced case), the differential equation is $mx'' + bx' + kx = 0$ and is simply a constant coefficient second order homogeneous equation we analyzed previously. However, now we have the restrictions: $m > 0, k > 0, b \geq 0$. The corresponding characteristic equation is $mr^2 + br + k = 0$ which has the solutions $r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = \frac{-b \pm \sqrt{d}}{2m}$ and so the behavior of the solution depends on the quantity $b^2 - 4mk$. Here is a short overview of the different cases depending on the damping constant b (for a fixed mass and spring):

- No damping ($b = 0$). In this case:

$$r = \pm \frac{\sqrt{-4mk}}{2m} = \pm \frac{2i\sqrt{m}\sqrt{k}}{2m} = \pm i\sqrt{\frac{k}{m}}$$

The solution is:

$$x(t) = C_1 \cos(w_0 t) + C_2 \sin(w_0 t) \quad \text{with} \quad w_0 = \sqrt{\frac{k}{m}}$$

known as the natural frequency. Notice that this can be written also as $x(t) = A\cos(w_0 t - \gamma)$ for a constant amplitude A and phase shift γ. The behavior is oscillatory for all time, with the amplitude of oscillations bounded as $|x(t)| \leq |A|$ for all time. Said in plan language: the block moves back and forth passing the equilibrium position ($x = 0$) infinitely many times.

- Underdamping ($b^2 - 4mk < 0$). When b is small then $b^2 - 4mk$ will be negative since $m, k > 0$. In this case, $r = \frac{-b}{2m} \pm i\frac{\sqrt{4mk-b^2}}{2m} = \alpha \pm i\beta$ with $\alpha = -\frac{b}{2m}$ and $\beta = \frac{\sqrt{-d}}{2m}$. The solution is:

$$x(t) = e^{-\frac{b}{2m}t}\left[C_1\cos(\beta t) + C_2\sin(\beta t)\right] = De^{-\frac{b}{2m}t}\cos(\beta t - \gamma)$$

Notice here that the behavior is oscillatory, but as $-\frac{b}{2m} < 0$, the amplitude of oscillations decreases (goes to zero) with increasing time.

- Critical damping ($b^2 - 4mk = 0$). In this case, $r = \frac{-b}{2m}$ is a single root of the quadratic characteristic equation. So we have the solution:

$$x(t) = C_1 e^{-\frac{b}{2m}t} + C_2 t e^{-\frac{b}{2m}t}$$

The behavior is clearly not oscillatory and the block will return to the equilibrium position in the shortest amount of time. (Formally $x(t) \to 0$ as $t \to \infty$, but $x(t)$ gets very close to zero very quickly). However, in doing so, the block can cross the t-axis (that is, pass through the equilibrium point $x = 0$) at most once.

- Overdamping ($b^2 - 4mk > 0$). In this case, both roots of the characteristic equation are real and negative:

$$r_1 = \frac{-b - \sqrt{d}}{2m} \quad \text{and} \quad r_2 = \frac{-b + \sqrt{d}}{2m}$$

Notice that $r_1, r_2 < 0$, since $\sqrt{d} = \sqrt{b^2 - 4mk} < \sqrt{b^2} = b$. The solution in this case is:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Behavior is similar to the critically damped case. There is no oscillation and the block will return quickly to equilibrium position (formally this takes place as $t \to \infty$ but it gets very close to zero quickly), but not as quickly as in the critically damped case (think of a lot of friction preventing the block from returning to the equilibrium position quickly). The block can also cross the equilibrium at most once (you can find the single possible crossing time by setting $x(t) = 0$ and solving for t).

Next, we discuss the forced case when $F(t) \neq 0$. The function $F(t)$ we will consider takes the form $F_0\cos(wt)$ where F_0 is some constant and w is some driving frequency. That is, in this case, the amplitude of the forcing is bounded for all time as $|F_0\cos(wt)| < |F_0|$ for all t. We will consider two cases in this setup: the undamped case $b = 0$ and the damped case $b > 0$:

- Forced, undamped case ($F(t) \neq 0$, $b = 0$). In this case, the equation is $mx'' + kx = F_0\cos(wt)$. The intrinsic (or so called natural) frequency of the system is given by $w_0 = \sqrt{\frac{k}{m}}$ as before. That's because the corresponding homogeneous equation $mx_h'' + kx_h = 0$ has characteristic equation $mr^2 + k = 0 \implies r = \pm i\sqrt{\frac{k}{m}}$ so that $x_h(t) = C_1\cos(w_0 t) + C_2\sin(w_0 t)$. When $w_0 \neq w$, we will propose the particular solution $x_p(t) = A\cos(wt) + B\sin(wt)$. In this case, then the whole solution $x(t) = x_h(t) + x_p(t)$ will be bounded for all time. The behavior of the solution will be oscillatory and the sum of two oscillations given by some combination of $\cos(w_0 t)$ and $\cos(wt)$.

When $w_0 = w$ (the natural frequency equals the forcing frequency), we will have the resonant case where the corresponding homogeneous equation solution (the solution to $mx_h'' + kx_h = 0$) has terms linearly dependent with the particular solution we propose given $f(t) = F_0\cos(wt)$. In this case, we need to choose $x_p(t) = t[C_1\cos(w_0 t) + C_2\sin(w_0 t)]$ which when evaluated will result in a general solution $x(t)$ whose magnitude $|x(t)|$ goes to ∞ as $t \to \infty$. The increasing amplitude of the resulting oscillations will cause the mechanical system to break. This case is known as resonant behavior.

- Forced, damped case ($F(t) \neq 0$, $b > 0$). In this case, the equation is $mx'' + bx' + kx = F_0\cos(wt)$. The homogeneous equation $mx_h'' + bx_h' + kx_h = 0$ has roots $r = \frac{-b\pm\sqrt{b^2-4mk}}{2m}$. Note that no matter what the value of $b^2 - 4mk$, when $b > 0$, $\lim_{t\to\infty} x_h(t) = 0$. That is, the homogeneous equation solution tends to zero. For this reason, the behavior of the system for large time depends entirely on the particular solution (known also in this case as the steady state solution). For the particular solution we can use $x_p(t) = A\cos(wt) + B\sin(wt)$. If we insert this and simplify, we can get the constants quoted in section 4.6 of the textbooks. Most importantly, the particular solution can then be written as $x_p(t) = \bar{A}\cos(wt-\gamma)$ where $\bar{A} = \sqrt{A^2 + B^2}$ and $\bar{A} = \frac{F_0}{\sqrt{m^2(w_0^2-w^2)^2+(bw)^2}} \leq \frac{F_0}{bw}$. Notice that the upper bound of the amplitude $\bar{A}$ is achieved precisely when $w_0 = w$, but in the damped case, the amplitude does not become unbounded with increasing time. However, if the damping level is low (b is small), then when the forcing frequency is equal to the intrinsic frequency, the amplitude of oscillations may indeed be very large and the mechanical system can still break. On the other hand, if the damping is large enough, then even when $w_0 = w$, the amplitude spike will be modest.

References:
http://ocw.mit.edu/courses/mathematics/18-03sc-differential-equations-fall-2011/unit-ii-second-order-constant-coefficient-linear-equations/, http://tutorial.math.lamar.edu/Classes/DE/Vibrations.aspx

3 Laplace Transforms

Textbook: Chapter 8.1,8.2

See attachment and:
http://tutorial.math.lamar.edu/Classes/DE/LaplaceIntro.aspx

1) $y'' - 4y' + 3y = 10\cos t$

$\Rightarrow r^2 - 4r + 3 = 0 \Rightarrow r = 1, 3$

$y_h(t) = c_1 e^t + c_2 e^{3t}$

Set $y_p(t) = A\cos(t) + B\sin(t)$

$\Rightarrow y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

plug into $L(y_p) = 10\cos(t)$

$-A\cos t - B\sin t + 4A\sin t - 4B\cos t$

$+ 3A\cos t + 3B\sin t =$

$= (4A + 2B)\sin t + (2A - 4B)\cos t = 10\cos(t)$

$\Rightarrow 4A + 2B = 0 \Rightarrow 2A = -B$

$2A - 4B = 10 \Rightarrow -B - 4B = -5B = 10$

so $B = -2$; $A = -\frac{B}{2} = +\frac{2}{2} = +1$

$\Rightarrow y_p(t) = +\cos t - 2\sin t$

$\Rightarrow y(t) = c_1 e^t + c_2 e^{3t} + \cos t - 2\sin t$

Note: "difficulty" if term in proposed y_p duplicates a term in y_h.

2) $y'' - 2y' + y = 3e^t$

$\Rightarrow r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$

$y_h(t) = c_1 e^t + c_2 t e^t$

Both e^t and te^t are in homogen. eqn solution. Thus, we must use as particular solution:

$$y_p(t) = At^2 e^t$$

If we use Ae^t or Ate^t, we will get $(N = 0)$ with $N \neq 0$, contradiction.

3) $y'' - 4y' + 4y = te^{2t}$

$te^{2t} = (1^{st} \text{ deg poly in } t) \times e^{2t}$

So generally we would set

$$y_p(t) = (At + B)e^{2t}$$

But in this case this won't work.

Notice that the homogeneous eqn solution : $y_h'' - 4y_h' + 4y_h = 0$

$\Rightarrow r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0$

$\Rightarrow r = 2$ (double root)

$\Rightarrow y_h(t) = c_1 e^{2t} + c_2 t e^{2t}$

$Ate^{2t} + Be^{2t}$ doesn't work

$At^2 e^{2t} + Bte^{2t}$ doesn't work

b/c te^{2t} appears in $y_h(t)$.

But $y_p(t) = At^3 e^{2t} + Bt^2 e^{2t}$ will work.

Note: what does it mean that te^{2t} appears in $y_h(t)$? It means $L(te^{2t}) = 0$, that it satisfies the homogeneous equation.

Example with more complicated rhs which is a product of functions:

$$y'' - 2y' - 3y = t^3 e^{5t} \cos(3t)$$

$$\Rightarrow r^2 - 2r - 3 = 0 = (r-3)(r+1)$$

$$y_h(t) = C_1 e^{3t} + C_2 e^{-t}$$

choose

$$y_p(t) = [At^3 + Bt^2 + Ct + D] e^{5t} \cos(3t) + [Et^3 + Ft^2 + Gt + H] e^{5t} \sin(3t)$$

Notes on Laplace Transforms

The Laplace transform of $f(t)$ is a function of a parameter s, defined via:

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t)\,dt$$

Linearity property: $a, b \in \mathbb{R}$ $\quad L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$

Some simple transforms:

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\,dt = \lim_{b\to\infty} \int_0^b e^{-(s-a)t}\,dt$$

$$= \lim_{b\to\infty} \left[\frac{e^{-(s-a)b}}{(s-a)} + \frac{1}{s-a} \right] = \frac{1}{s-a} \quad \text{for } s>a$$

Notice $s>a$ is necessary for $(s-a)>a$ so that the limit integral above converges.

Note: $L[1] = L[e^0] = \dfrac{1}{s}$

$$L[e^{ikt}] = L[\cos(kt) + i\sin(kt)] = L[\cos(kt)] + i\,L[\sin(kt)]$$

Since $L[e^{at}] = \dfrac{1}{s-a}$ it follows that:

$$L[e^{ikt}] = \frac{1}{s-ik}\,\frac{s+ik}{s+ik} = \frac{s+ik}{s^2-(ik)^2} = \frac{s+ik}{s^2+k^2}$$

since $i^2 = -1$.

$$\Rightarrow L[e^{ikt}] = \frac{s}{s^2+k^2} + i\,\frac{k}{s^2+k^2}$$

$$= L[\cos(kt)] + i\,L[\sin(kt)]$$

It follows that: $L[\cos(kt)] = \dfrac{s}{s^2+k^2}$; $L[\sin(kt)] = \dfrac{k}{s^2+k^2}$

Next, we calculate $L[t^n]$ for some integer $n \geq 0$.

$$u = t^n \ ; \ dv = e^{-st}dt \quad \Rightarrow v = -\frac{e^{-st}}{s} \quad du = nt^{n-1}dt$$

$$L[t^n] = \int_0^\infty e^{-st}\,t^n\,dt =$$

$$= \underbrace{-\frac{t^n e^{-st}}{s}\Big|_0^\infty}_{uv|_0^\infty} + \underbrace{\frac{n}{s}\int_0^\infty e^{-st}\,t^{n-1}\,dt}_{-\int_0^\infty v\,du}$$

$$= \frac{n}{s}L[t^{n-1}] = \frac{n}{s}\left(\frac{n-1}{s}\right)L[t^{n-2}] = \dots = \frac{n!}{s^n}L[1]$$

Since $L[1] = \frac{1}{s}$, we get:

$$L[t^n] = \frac{n!}{s^{n+1}}$$

Shift of transform:

$$L[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t)\,dt =$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$$

where $F(s) = L[f(t)]$. Thus:

$$L[\cos(kt)] = \frac{s}{s^2+k^2} \Rightarrow L[e^{at}\cos(kt)] = \frac{s-a}{(s-a)^2+k^2}$$

$$L[\sin(kt)] = \frac{k}{s^2+k^2} \Rightarrow L[e^{at}\sin(kt)] = \frac{k}{(s-a)^2+k^2}$$

Transform of derivatives:

$$L[f(t)] = \int_0^\infty e^{-st}f(t)\,dt$$

$$L[f'(t)] = \int_0^\infty e^{-st}f'(t)\,dt = [e^{-st}f(t)]\Big|_0^\infty - \int_0^\infty f(t)[-se^{-st}dt]$$

$(u = e^{-st};\ dv = f'(t)dt \Rightarrow v = f(t) \Rightarrow du = -se^{-st}dt$

Since $\lim_{b\to\infty} \left[e^{-st} f(t)\right]\Big|_{t=b} = 0$, we get:

$$L[f'(t)] = 0 - f(0) + sL[f(t)] = sL[f(t)] - f(0)$$

Using this formula, we get:

$$L[f''(t)] = sL[f'(t)] - f'(0) =$$

$$= s\left[sL[f(t)] - f(0)\right] - f'(0)$$

$$= s^2 L[f(t)] - sf(0) - f'(0)$$

<u>Ex</u> Solve the IVP:

$$y'' + y' + y = 1 \; ; \quad y(0) = 0, \; y'(0) = 0$$

$$L[y''] + L[y'] + L[y] = L[1] = \frac{1}{s}$$

$$\Rightarrow \underbrace{s^2 Y(s) - s\,y(0) - y'(0)}_{L[y'']} + \underbrace{sY(s) - y(0)}_{L[y']} + Y(s) = \frac{1}{s}$$

$$\Rightarrow s^2 Y(s) + sY(s) + Y(s) = \frac{1}{s} \quad \text{using the ICs.}$$

we can factor and solve for $Y(s) = L[y(t)]$:

$$Y(s)[s^2+s+1] = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s(s^2+s+1)}$$

All that remains is to invert $Y(s)$ to get $y(t)$.

Use partial fraction decomposition:

$$\frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$$\Rightarrow 1 = A(s^2+s+1) + [Bs+C]s$$

use $s=0 \Rightarrow 1 = A(1) \Rightarrow A=1$

$$\Rightarrow 1 = s^2+s+1 + Bs^2+Cs \Rightarrow B=-1, \; C=-1$$

$$\Rightarrow \frac{1}{s(s^2+s+1)} = \frac{1}{s} - \frac{(s+1)}{s^2+s+1} = \frac{1}{s} - \frac{(s+1)}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

To make use of formulas for $L[e^{at}\cos(\kappa t)]$, $L[e^{at}\sin(\kappa t)]$, we further manipulate the fraction:

$$\frac{1}{s(s^2+s+1)} = \frac{1}{s} - \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\frac{3}{4}} - \frac{\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

Finally, we multiply the last term by $\frac{\sqrt{3}}{\sqrt{3}}$:

$$Y(s) = \frac{1}{s(s^2+s+1)} = \frac{1}{s} - \frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}/2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

This is done so that $\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$, consistent with

$$L[e^{at}\sin(\kappa t)] = \frac{\kappa}{(s-a)^2+\kappa^2}$$

Thus, we find that:

$$y(t) = L^{-1}[Y(s)] = 1 - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right) - e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

Ex | Solve: $y'' + 4y = 4t \;;\; y(0) = 1,\ y'(0) = 5$

$$\Rightarrow L[y''] + 4L[y] = 4L[t] = \frac{4}{s^2}$$

$$\Rightarrow [s^2 L[y] - sy(0) - y'(0)] + 4L[y] = \frac{4}{s^2}$$

$$\Rightarrow s^2 L[y] - s - 5 + 4L[y] = \frac{4}{s^2}$$

$$\Rightarrow (s^2+4)L(y) = s + 5 + \frac{4}{s^2}$$

Thus, $$L(y) = \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{4}{s^2(s^2+4)}$$

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{B}{s^2+4} \Rightarrow 1 = A(s^2+4) + Bs^2$$

$$s=0 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4} \Rightarrow 1 = \frac{1}{4}s^2 + 1 + Bs^2 \Rightarrow B = -\frac{1}{4}$$

$$\Rightarrow L(y) = \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{1}{s^2} - \frac{1}{s^2+4}$$

$$= \frac{1}{s^2} + \frac{s}{s^2+4} + \underbrace{\frac{4}{s^2+4}}_{=2\frac{2}{s^2+2^2}} = Y(s)$$

$$\Rightarrow y(t) = L^{-1}[Y(s)] = t + \cos(2t) + 2\sin(2t)$$

Multiplier rule for transform of $tf(t)$:

$$L[tf(t)] = \int_0^\infty t e^{-st} f(t)\,dt =$$

$$= -\int_0^\infty \frac{d}{ds}\left(e^{-st}\right) f(t)\,dt = -\frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= -\frac{d}{ds} F(s) = -F'(s) = -\frac{d}{ds} L[f(t)]$$

Ex) Since $L[e^{at}] = \frac{1}{s-a}$ we find:

$$L[te^{at}] = -\frac{d}{ds}\left[L[e^{at}]\right] = -\frac{d}{ds}\left[\frac{1}{s-a}\right] =$$

$$= -\left[-(s-a)^{-2}\right] = \frac{1}{(s-a)^2}$$

Ex) $y'' + 2y' + y = 1, \quad y(0) = 2, \; y'(0) = -2$

$$\Rightarrow \left(s^2 L[y] - sy(0) - y'(0)\right) + 2\left(sL[y] - y(0)\right) + L[y]$$

$$= L[1] = \frac{1}{s}$$

$$\Rightarrow (s^2 + 2s + 1)Y(s) - (s+2)y(0) - y'(0) = \frac{1}{s}$$

Plugging in the IC's, we find:

$$(s^2+2s+1)Y(s) - 2(s+2) + 2 = \frac{1}{s}$$

$$\Rightarrow (s^2+2s+1)Y(s) = \frac{1}{s} + 2s + 2 \Rightarrow s(s+1)^2 Y(s) = 1 + 2s^2 + 2s$$

$$\Rightarrow Y(s) = \frac{2s^2+2s+1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$= \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2} = \frac{A(s+1)^2 + s[B(s+1)+C]}{s(s+1)^2}$$

Upon expanding, we find:

$$2s^2+2s+1 = (A+B)s^2 + (2A+B+C)s + A$$

$$\Rightarrow A=1,\ B=1,\ 2+1+C=2 \Rightarrow C=-1$$

$$\Rightarrow Y(s) = \frac{2s^2+2s+1}{s(s+1)^2} = \frac{1}{s} + \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$y(t) = L^{-1}[Y(s)] = 1 + e^{-t} - te^{-t}$$

Notice we used the result $L[te^{at}] = \frac{1}{(s-a)^2}$

so that $L^{-1}\left[\frac{1}{(s-a)^2}\right] = te^{at}$.

Derivatives of inverse transform:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = \int_0^\infty f(t) e^{-st} (-t)\,dt = \int_0^\infty [-t f(t)] e^{-st}\,dt$$

$$\Rightarrow L^{-1}[F'(s)] = -t f(t) = -t L^{-1}[F(s)]$$

Notice that this can be derived also from the multiplier rule $L[t f(t)] = -\frac{d}{ds} L[f(t)] = -F'(s)$.

Ex) $x'' + 9x = \cos(3t)$, $x(0)=1$, $x'(0)=-1$

This corresponds to forced shm, no damping.

Notice that $w_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{9}{1}} = 3$ and $w=3$

so $w = w_0$ (this is the resonant case, no damping).

Hence, we expect $|x(t)| \to \infty$ as $t \to \infty$.

$$\Rightarrow s^2 L[x] - s(\underbrace{x(0)}_{=1}) - \underbrace{x'(0)}_{=-1} + 9L[x] = L[\cos(3t)]$$

$$= \frac{s}{s^2+9}$$

$$\Rightarrow s^2 X(s) - s + 1 + 9X(s) = \frac{s}{s^2+9} \Rightarrow X(s)\left[s^2+9\right] = s-1+\frac{s}{s^2+9}$$

$$\Rightarrow X(s) = \frac{s-1}{(s^2+9)} + \frac{s}{(s^2+9)^2}$$

$$= \frac{s}{s^2+9} - \frac{1}{3}\frac{3}{s^2+3^2} + \frac{s}{(s^2+9)^2}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] = \cos(3t) \; ; \; \mathcal{L}^{-1}\left[\frac{1}{3}\frac{3}{s^2+3^2}\right] = \frac{1}{3}\sin(3t)$$

For the last term note that:

$$\frac{d}{ds}\left[\frac{1}{s^2+9}\right] = -(s^2+9)^{-2}(2s)$$

$$\Rightarrow \frac{s}{(s^2+9)^2} = -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s^2+9}\right] = -\frac{1}{2}F'(s)$$

$$\mathcal{L}^{-1}\left[F'(s)\right] = -t\,\mathcal{L}^{-1}\left[F(s)\right] = -t\,\mathcal{L}^{-1}\left[\frac{1}{3}\frac{3}{s^2+3^2}\right] = -\frac{t}{3}\sin(3t)$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{(s^2+9)^2}\right] = \mathcal{L}^{-1}\left[-\frac{1}{2}F'(s)\right] = -\frac{1}{2}\mathcal{L}^{-1}\left[F'(s)\right] = \frac{t}{6}\sin(3t)$$

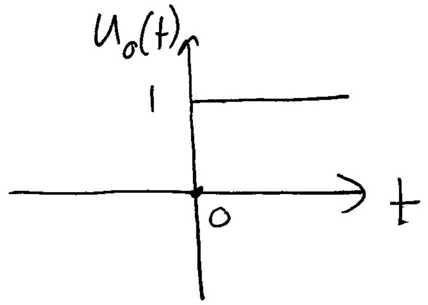
$$\Rightarrow x(t) = \cos(3t) - \frac{1}{3}\sin(3t) + \frac{t}{6}\sin(3t)$$

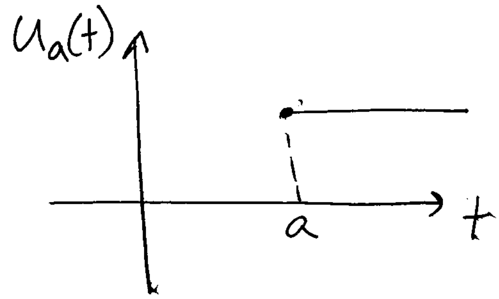
Notice: due to last term, $|x(t)| \to \infty$ as $t \to \infty$.

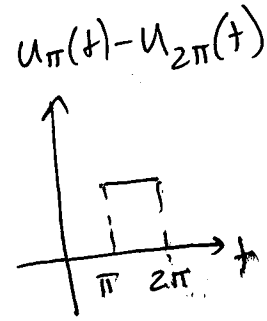
Step functions

$$u_a(t) = \text{step}(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases} \quad ; a \geq 0$$

where $u_0(t) = \text{step}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$



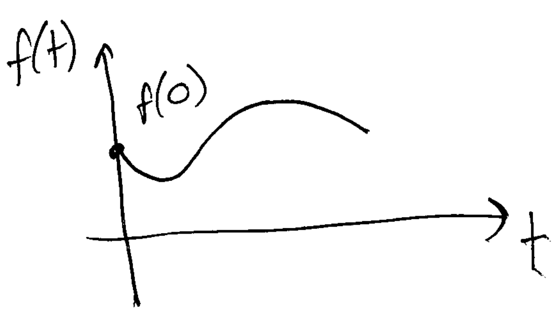


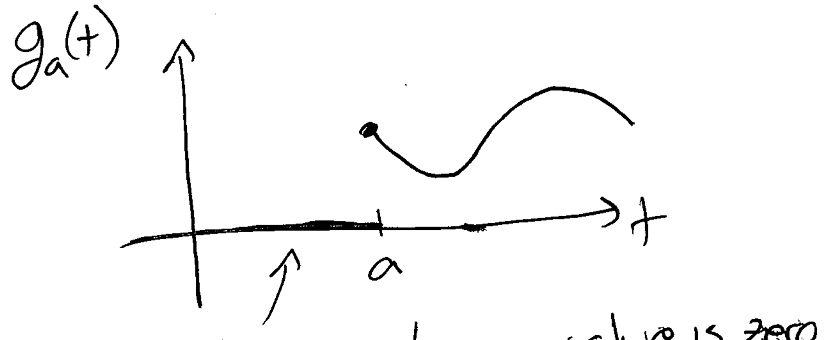


Delayed translation function

For a given function $f(t)$, define:

$$g_a(t) = u_a(t)\, f(t-a) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \end{cases}$$





here for $t<a$, value is zero.

We now calculate Laplace transforms of the step and delayed functions:

$$L[u_a(t)] = L[\text{step}(t-a)] = \int_0^\infty e^{-st} u_a(t)\, dt$$

$$= \int_a^\infty e^{-st} dt = \frac{e^{-as}}{s}, \quad s>0$$

$$L[g_a(t)] = L[u_a(t) f(t-a)] = \int_0^\infty e^{-st} u_a(t) f(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

Next, let $\varepsilon = t-a \Rightarrow d\varepsilon = dt \Rightarrow a = t-\varepsilon \,;\, t = \varepsilon + a$

$$\Rightarrow L[g_a(t)] = \int_0^\infty e^{-(\varepsilon+a)s} f(\varepsilon)\, d\varepsilon = e^{-as} \int_0^\infty e^{-\varepsilon s} f(\varepsilon)\, d\varepsilon$$

$$= e^{-as} F(s)$$

Hence, $L^{-1}\{e^{-as} F(s)\} = u_a(t) f(t-a)$

As an example, find the inverse transform of

$$F(s) = \frac{1-e^{-2s}}{s^2} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$$

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$

$$= t - u_2(t)(t-2)$$

$$= t - \text{step}(t-2)(t-2)$$

Note that $f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 2, & t \ge 2 \end{cases}$

Ex) Let $g(t) = \begin{cases} 0, & t < \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}), & t \ge \frac{\pi}{4} \end{cases} = u_{\frac{\pi}{4}}(t)\cos(t - \frac{\pi}{4})$

Then, $L[g(t)] = L[u_{\frac{\pi}{4}}(t)\cos(t - \frac{\pi}{4})] = e^{-\frac{\pi}{4}s} L[\cos(t)]$

$$= e^{-\frac{\pi}{4}s}\frac{s}{s^2+1}$$

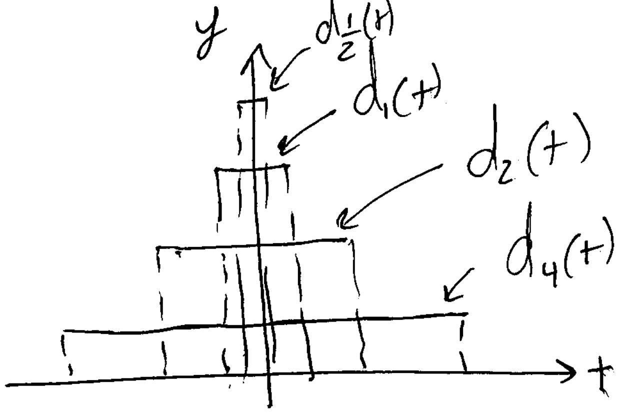
Impulse and delta functions

Let $d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & t \le -\tau,\ t \ge \tau \end{cases}$

Note that $\lim_{\tau \to 0} d_\tau(t) = 0$ for $t \ne 0$.

Also,

$$\int_{-\infty}^{\infty} d_\tau(t)\,dt = \int_{-\tau}^{\tau} \frac{1}{2\tau}\,dt = \frac{1}{2\tau}[\tau+\tau] = 1.$$



(not to scale)

As $\tau \to 0$, this approximates a unit impulse function (for example, an explosion), which imparts an impulse of unit magnitude at $t=0$.

$$\delta(t) = \lim_{\tau\to 0} d_\tau(t) \quad \text{(delta function)}$$

$$\delta(t-t_0) = \lim_{\tau\to 0} d_\tau(t-t_0) \quad \left(\begin{array}{l}\text{shifted delta function}\\ \text{eg. explosion at } t=t_0\end{array}\right\}$$

$$\Rightarrow L[\delta(t-t_0)] = \lim_{\tau\to 0} L[d_\tau(t-t_0)]$$

We will show that the limit is $1\cdot e^{-st_0}$.

$$L[d_\tau(t-t_0)] = \int_0^{\infty} e^{-st} d_\tau(t-t_0)\,dt = \int_{t_0-\tau}^{t_0+\tau} e^{-st} d_\tau(t-t_0)\,dt$$

$$\Rightarrow L[d_\tau(t-t_0)] = \int_{t_0-\tau}^{t_0+\tau} e^{-st} d_\tau(t-t_0)\,dt$$

$$= \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}dt = \frac{1}{2\tau}\,\frac{e^{-st}}{s}\Bigg|_{t=t_0-\tau}^{t=t_0+\tau}$$

$$= \frac{1}{2\tau}\,\frac{1}{s}\left[e^{-s(t_0+\tau)} - e^{-s(t_0-\tau)}\right]$$

$$= \frac{1}{2s\tau}\,e^{-st_0}\left(e^{s\tau}-e^{-s\tau}\right)$$

Recall the hyperbolic functions:

$\sinh(z) = \frac{1}{2}(e^{z}-e^{-z})$ and $\cosh(z) = \frac{1}{2}(e^{z}+e^{-z})$

$$\Rightarrow L[d_\tau(t-t_0)] = e^{-st_0}\,\frac{\sinh(s\tau)}{s\tau}$$

$$\Rightarrow \lim_{\tau\to 0} L[d_\tau(t-t_0)] = e^{-st_0}\lim_{\tau\to 0}\frac{\sinh(s\tau)}{s\tau}$$

$$= e^{-st_0}\lim_{\tau\to 0}\frac{s\cosh(s\tau)}{s} = e^{-st_0}$$

Thus, we have: $L[\delta(t)] = 1$

$L[\delta(t-t_0)] = e^{-st_0}$

Ex) $2y'' + y' + 2y = \delta(t-5)$ (delta fcn source at $t=5$ unit magnitude)

$y(0)=0,\ y'(0)=0$

$\Rightarrow (2s^2+s+2)Y(s) = e^{-5s}$

$\Rightarrow Y(s) = \dfrac{e^{-5s}}{2s^2+s+2} = \dfrac{e^{-5s}}{2}\,\dfrac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}$

$\Rightarrow L^{-1}\left\{\dfrac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}\right\} = \dfrac{4}{\sqrt{15}}e^{-t/4}\sin\left(\dfrac{\sqrt{15}}{4}t\right)$ (note: $\left(\frac{\sqrt{15}}{4}\right)^2 = \frac{15}{16}$)

$\Rightarrow y(t) = L^{-1}\{Y(s)\} = \dfrac{2}{\sqrt{15}}\underbrace{u_5(t)}_{\text{step}(t-5)}e^{-(t-5)/4}\sin\left(\dfrac{\sqrt{15}}{4}\right)(t-5)$

$= \begin{cases} 0, & t<5 \\ \dfrac{2}{\sqrt{15}}e^{-(t-5)/4}\sin\dfrac{\sqrt{15}}{4}(t-5), & t\ge 5 \end{cases}$

Notice: As ICs are zero, no motion until $t=5$ (explosion time)

Ex) (8.3.52) $x'' + x = \delta(t-2\pi)$, $x(0)=1$, $x'(0)=0$

$[s^2X(s) - s\,x(0) - x'(0)] + X(s) = e^{-2\pi s}$

Now $L^{-1}\left\{\dfrac{s}{s^2+1}\right\} = \cos(t)$

$L^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin(t)$

$\Rightarrow (s^2+1)X(s) - s = e^{-2\pi/s}$

$\Rightarrow X(s) = \dfrac{s}{s^2+1} + e^{-2\pi/s}\left(\dfrac{1}{s^2+1}\right) \Rightarrow$ ~~[illegible]~~

$\Rightarrow x(t) = \cos(t) + \underbrace{\sin(t-2\pi)\ \text{step}(t-2\pi)}$

$\Rightarrow$ motion due to IC up to $t=2\pi$, then also motion due to source $u_{2\pi}(t)\,g(t-2\pi)$

Ex) $x'(t) = \delta(t) - \delta(t-1)$, $x(0) = 0$

"explosion at $t=0$, followed by impulse in opposite direction a time $t=1$"; after $t=1$ is reached, impulses "cancel" each other out.

$$\Rightarrow sX(s) - \cancel{x}(0) = L[\delta(t)] - L[\delta(t-1)]$$
$$= 1 - e^{-s}$$

$$\Rightarrow sX(s) = 1 - e^{-s} \quad \Rightarrow X(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\Rightarrow x(t) = L^{-1}\{X(s)\} = 1 - \text{step}(t-1)$$

since $L[u_1(t)] = L[\text{step}(t-1)] = \dfrac{e^{-s}}{s}$

note: $x(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$ motion ceases after second impulse cancels out the first.

Convolutions

$$L[f] \cdot L[g] = L[f * g](s)$$

$$\Rightarrow L^{-1}\Big[L[f] \cdot L[g]\Big] = f * g(t) = \int_0^t f(t-x) g(x)\, dx$$

Ex) Calculate $L^{-1}\left[\dfrac{1}{s^2(s^2+1)}\right] = L^{-1}\left[\underbrace{\frac{1}{s^2}}_{L[t]} \cdot \underbrace{\frac{1}{s^2+1}}_{L[\sin(t)]}\right]$

Note that $\frac{1}{s^2} = L[t]$; $\frac{1}{s^2+1} = L[\sin(t)]$

$$\Rightarrow L^{-1}\left[\frac{1}{s^2}\frac{1}{s^2+1}\right] = \int_0^t (t-x)\sin x\,dx = t-\sin t$$

$$= \int_0^t t\sin x\,dx - \int_0^t x\sin x\,dx$$

$$= -t\cos x\Big|_0^t - (\sin x - x\cos x)\Big|_0^t$$

$$= -t[\cos(t)-1] - [\sin t - t\cos(t) - 0]$$

$$= -t\cos(t) + t - \sin(t) + t\cos(t) = t - \sin(t)$$

Ex) $1 * 1$

$$(f*g)(t) = \int_0^t f(t-w)\,g(w)\,dw$$ (note: w "dummy" variable can use x too).

$$\Rightarrow 1*1 = \int_0^t 1\cdot 1\,dt = t$$

Ex) $1*1*1 = 1*t = \int_0^t 1\cdot w\,dw = \frac{w^2}{2}\Big|_0^t = \frac{1}{2}t^2$

Ex) $t*t = \int_0^t (t-w)\,w\,dw = \int_0^t (tw - w^2)\,dw$

$$= t\frac{w^2}{2}\Big|_0^t - \frac{w^3}{3}\Big|_0^t = \frac{t^3}{2} - \frac{t^3}{3} = \frac{t^3}{6}$$

Ex) compute $e^{at} * e^{-at} = f(t) * g(t)$

$$= \int_0^t f(t-w)\, g(w)\, dw = \int_0^t e^{a(t-w)} e^{-aw}\, dw$$

$$= \int_0^t e^{at-2aw}\, dw = e^{at} \int_0^t e^{-2aw} dw = e^{at} \frac{e^{-2aw}}{-2a} \Big|_0^t$$

$$= e^{at}\left[-\frac{e^{-2at}}{+2a} + \frac{e^0}{2a} \right] = \frac{e^{at}}{2a} - \frac{e^{-at}}{2a}$$

$$= \frac{1}{a} \sinh(at)$$

Transfer and impulse response functions

$y' + ay = f(t)$ has solution:

$$y(t) = \int_0^t e^{-a(t-w)} f(w)\, dw = e^{-at} * f(t)$$

$x' + ax = f(t)\ ;\ x(0) = 0$

$sX(s) + aX(s) = L[f]$

$\Rightarrow X(s) = \frac{1}{s+a} L[f] \Rightarrow$

transfer function $= \frac{1}{s+a}$

impulse function $= L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

The output of a linear system is the convolution of the impulse response function and transfer function.

$$\Rightarrow x(t) = I(t) * f(t) = \int_0^t e^{-a(t-w)} f(t)\,dw$$

General eq:

$$ay'' + by' + cy = f(t), \quad y(0) = y'(0) = 0$$

$$aY(s)s^2 + bsY(s) + cY(s) = L[f(t)]$$

$$\Rightarrow Y(s) = \frac{1}{as^2+bs+c} L[f(t)] = \overset{= L[h](s)}{\text{function of } s} \times \overset{= L[f](s)}{\text{another fcn of } s}$$

$$\Rightarrow L[y] = \overset{\text{transfer fcn}}{L[h]} \cdot L[f] \Rightarrow y(t) = L^{-1}\left[L[h] \cdot L[f]\right] = h * f(t)$$

$$\Rightarrow y(t) = h(t) * f(t) = \int_0^t h(t-w)\, f(w)\,dw$$

Ex] $x'' + x = f(t), \quad x(0) = x'(0) = 0$

$$\Rightarrow s^2 X(s) + X(s) = L[f] \Rightarrow X(s) = \frac{1}{s^2+1} L[f]$$

transfer fcn: $\frac{1}{s^2+1}$ $\Rightarrow$ impulse funct: $I(t) = \sin t = L^{-1}\left[\frac{1}{s^2+1}\right]$

$$x(t) = I(t) * f(t) = \int_0^t \sin(t-w)\, f(w)\,dw$$