Covered Material and Resources for Part4 (Exam 3 - Final)

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1 Eigenvalues/Eigenvectors

Textbook: Section 5.3

Any linear operator T which maps vectors from $\mathbb{R}^m$ to $\mathbb{R}^n$ can be represented by a matrix $A \in \mathbb{R}^{m\times n}$ such that for $w \in \mathbb{R}^m$, we have $z = T(w) = Aw \in \mathbb{R}^n$.

For square matrices $A \in \mathbb{R}^{n\times n}$, some special vectors v will be such that Av is a scalar multiple of v (the scalar multiple being λv for some $\lambda \in \mathbb{R}$). We say that nonzero vectors v and scalars λ satisfying $Av = \lambda v$ form an eigenvector/eigenvalue pair. Notice that $Av = \lambda v \implies (A - \lambda I)v = 0$. The system of equations $(A - \lambda I)v = 0$ has nonzero solutions only when $(A - \lambda I)$ is not invertible and this happens when $\det(A - \lambda I) = 0$. Hence, λ is an eigenvalue of A if and only if it satisfies the polynomial equation:

$$\det(A - \lambda I) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \dots c_1\lambda + c_0 = 0$$

Notice that the above expansion implies property (1) below. For if, $\lambda = 0$ is an eigenvalue of A then $\det(A) = c_0 = 0$ and so A is not invertible. Another property concern powers of the matrix A. Suppose $Av = \lambda v$ (that is, A has eigenvector v and eigenvalue λ). Then: $A^2v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v$. By induction, we arrive at property (2) below. Also notice that when A^{-1} does exist we have $A^{-1}(A - \lambda I)v = 0 = (I - \lambda A^{-1})v = 0 = (A^{-1} - \frac{1}{\lambda}I)v = 0$. This leads to property (3) below. A and A^T satisfy the same characteristic equation, hence have the same eigenvalues. That's because $\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \det((A - \lambda I)^T) = \det(A - \lambda I)$. This implies property (4). Eigenvalues can either be simple or repeated, depending on the roots of the characteristic equation $\det(A - \lambda I) = 0$. If an eigenvalue is a repeated root of multiplicity M, then property (5) applies. So for example, if A is 3×3 and has eigenvalue λ of multiplicity 3, then there may just be one linearly independent eigenvector corresponding to this eigenvalue or up to 3 depending on A. For complex eigenvalues, they and their corresponding eigenvectors always occur in conjugate pairs, as stated in property (6).

- (1) A square matrix A is invertible (when $\det(A) \neq 0$) if and only if $\lambda = 0$ is NOT an eigenvalue of A.
- (2) For $k > 0$, if A has eigenvector v and eigenvalue λ, then A^k has eigenvalue λ^k and same corresponding eigenvector v. That is, $A^k v = \lambda^k v$.

- (3) When A^{-1} exists (A does not contain a zero eigenvalue) then if $Av = \lambda v$ then $A^{-1}v = \frac{1}{\lambda}v$ (that is, $\frac{1}{\lambda}$ is an eigenvalue of A with the same eigenvector v).
- (4) A and A^T have the same eigenvalues, but usually different eigenvectors.
- (5) If matrix A has eigenvalue λ with multiplicity M then there exists between 1 and M linearly independent eigenvectors corresponding to this eigenvalue.
- (6) If matrix A has complex eigenvalue $\lambda_1 = \alpha + i\beta$ then it also has the eigenvalue $\lambda_2 = \alpha - i\beta$ which is the complex conjugate of λ_1. If v_1 is an eigenvector corresponding to λ_1 then v_2 is an eigenvector corresponding to λ_2 where v_2 is complex conjugate of v_1.

For example, consider the matrix:

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

It follows that:

$$\begin{aligned} \det(A - \lambda I) = \left| \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} \right| &= (2-\lambda) \left| \begin{bmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} \right| = (2-\lambda)\left[(-\lambda)(3-\lambda)+2\right] \\ &= (2-\lambda)[\lambda^2 - 3\lambda + 2] = (2-\lambda)(\lambda-2)(\lambda-1) = 0 \end{aligned}$$

It follows that $\lambda_{1,2} = 2$ is an eigenvalue of multiplicity 2 and $\lambda_3 = 1$ is a simple eigenvalue. There can be one or two linearly independent eigenvectors corresponding to $\lambda_{1,2}$ and there must be one linearly independent eigenvector corresponding to λ_3. We solve $(A-2I)v_{(1,2)} = 0$, where $v_{1,2} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

Omitting the last all zero column we row reduce:

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $v_1 + v_3 = 0 \implies v_1 = -v_3$ and v_2 can take any value. Set $v_2 = \alpha$ and $v_3 = \beta$, then $v_1 = -\beta$ and so:

$$v_{1,2} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, in this case there are two linearly independent eigenvectors for the repeated eigenvalue $\lambda = 2$ and they are $v_{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $v_{(2)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (or any multiple of these, for example, we can scale these to have unit norm). For the simple eigenvalue $\lambda = 1$, $(A - I)v_{(3)} = 0$ gives:

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which gives the equations $v_1 + v_2 + v_3 = 0$ and $-v_2 + v_3 = 0 \implies v_3 = v_2$. Letting $v_3 = \alpha$, we get $v_1 = -v_2 - v_3 = -2\alpha$, $v_2 = \alpha$, $v_3 = \alpha$:

$$v_{(3)} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2\alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

so the third eigenvector is $v_{(3)} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Next, consider the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \implies |(A - \lambda I)| = \left| \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \right| = \lambda^2 + 1 = 0 \implies \lambda = \pm i$$

We choose $\lambda_1 = i$ and compute $v_{(1)}$. The second eignevector is then the complex conjugate of the first. We get $(A - iI)v_{(1)} = 0$:

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $\det(A - \lambda I) = 0$, the rank of the above matrix is one and any one of the equations can be used (the other is just a scalar multiple of the first and gives the same information). Thus, we get $-iv_1 + v_2 = 0 \implies v_1 = 1, v_2 = i$ so that $v_{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$. This immediately implies that for $\lambda_2 = -i$, the eigenvector is $v_{(1)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Notice that an eigenspace for a given eigenvalue λ is a set given as the span (all possible linear combinations) of the corresponding eigenvectors. The dimension of the eigenspace is simply the number of eigenvectors corresponding to the eigenvalue, see the example below.

Ex) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Find basis for eigenspace of A.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$-1 \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ -\lambda & 1 \end{vmatrix}$$ (along 1^{st} column)

$$= -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda)$$ (need to factor this to get roots)

$$= -\lambda(\lambda+1)(\lambda-1) + (\lambda+1) + (\lambda+1)$$

$$= (\lambda+1)(-\lambda(\lambda-1) + 1 + 1) = (\lambda+1)(-\lambda^2 + \lambda + 2)$$

$$= -(\lambda+1)(\lambda^2 - \lambda - 2) = -(\lambda+1)(\lambda-2)(\lambda+1)$$

Thus, $\lambda_{1,2} = 1$; $\lambda_3 = 2$

$\lambda_{1,2} = -1$ is of multiplicity 2. λ_3 is 'simple' eval.

(A) we analyze eigenspace of $\lambda_{1,2} = -1$

$\Rightarrow$ Plug into $(A - \lambda_1 I)\vec{X}_{(1)} = \vec{0}$; solve for $\vec{X}_{(1)}$

$$(A - \lambda_{(1)} I)\vec{x}_{(1)} = \vec{0} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x_1 + x_2 + x_3 = 0 \quad \Rightarrow$ set $x_2 = p,\ x_3 = q$

then $x_1 = -x_2 - x_3 = -p - q$

$$\Rightarrow \vec{x}_{(1)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -p-q \\ p \\ q \end{pmatrix} = p\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + q\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\underset{\{\lambda_1 = -1\}}{\text{eigenspace}} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\underline{\text{basis}} \text{ for } \underset{\{\lambda_1 = -1\}}{\text{eigenspace}} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

= two linearly indep. eigenvectors corresponding to eigenvalue $\lambda_1 = -1$.

(B) we analyze eigenspace of $\lambda_3 = 2$

$$(A - \lambda_{(3)} I)\vec{x}_{(3)} = \vec{0} \Rightarrow \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right)$$

$$\rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \end{pmatrix}$$

~~clearly these are linearly dependent since~~ [illegible]

Recall that $\det(A - \lambda_{(3)} I) = 0$ so $(A - \lambda_{(3)} I)$ is not full rank.

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

have to at least reduce to this step to easily write sol.

$$x_1 - \frac{1}{2} x_2 - \frac{1}{2} x_3 = 0$$

$$-\frac{3}{2} x_2 + \frac{3}{2} x_3 = 0 \Rightarrow x_2 = x_3 = s$$

$$x_1 = \frac{1}{2} x_2 + \frac{1}{2} x_3 = \frac{1}{2} s + \frac{1}{2} s = s = x_2 = x_3$$

$$\Rightarrow \vec{x}_{(3)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s \\ s \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underset{\{\lambda_3=2\}}{\text{eigenspace}} = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

$$\underset{\{\lambda_3=2\}}{\text{basis for eigenspace}} = \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

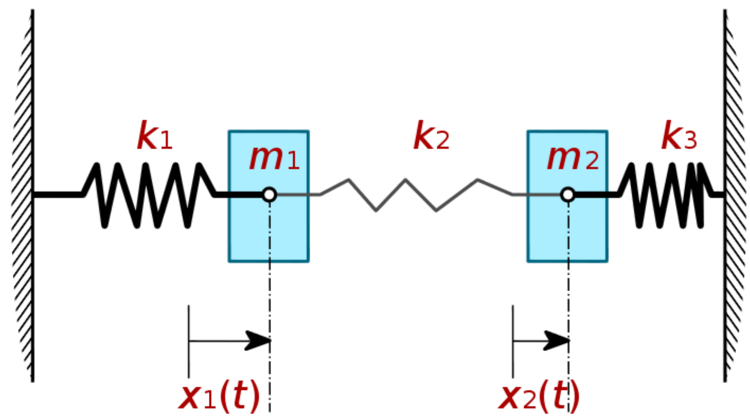
so eigenvalues and eigenvectors are:

$$\underbrace{\lambda_{1,2}=-1}_{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}}, \quad \underbrace{\lambda_3=2}_{\begin{pmatrix}1\\1\\1\end{pmatrix}};$$

Reference:
http://tutorial.math.lamar.edu/Classes/DE/LA_Eigen.aspx

2 Systems of ODEs and their solutions (Textbook: Section 6.1 - 6.3)

Systems of differential equations can be motivated by the following three spring two mass problem:



This setup is interesting because with the introduction of an external force, it can be used to model a simple tuned mass damper. The idea is to 'transfer' the vibrations of one big mass (a bridge or building) to a smaller mass so that the bigger amplitude vibrations are passed on to the smaller mass and the bigger mass remains relatively stable, even with a force (e.g. strong wind) acting on it. There are three springs with stiffness constants k_1, k_2, k_3 and two blocks of masses m_1 and m_2. The time dependent quantities $x_1(t)$ and $x_2(t)$ are the displacements of the masses from their equilibrium positions. For the simplest possible setup, we assume no frictional forces so that $b = 0$ and also that no external forces act on the blocks so the equations are homogeneous. The analysis is very similar with one or both of these included. To find the equations of motion, we analyze separately the forces acting on each mass. We may assume $x_1(t)$ and $x_2(t)$ are both to the right of the equilibrium position - that is both blocks are pushed against the springs k_1 and k_2 to the right and their equilibrium position is to their left (if they are pushed to the left or one pushed to the right and one to the left then both or one of $x_1(t), x_2(t)$ will be negative and the equations below will still hold).

- Block 1 (mass m_1): When $x_1(t)$ and $x_2(t)$ are both to the right of the equilibrium position, a force $-k_1x_1$ acts to oppose the displacement of m_1 to the left. Similarly, on the right, spring k_2 opposes the motion of the block and acts to the left giving component $-k_2x_1$. However, since block 2 (mass m_2) is also displaced to the right by amount x_2 it causes the spring k_2 to stretch towards m_2 via the force k_2x_2 (proportional to how much m_2 is moved - in opposing the motion of m_2 the spring exerts an equal forced push on m_1). Hence, the net force on block m_1 is:

$$F_1 = m_1\frac{d^2x_1}{dt^2} = -k_1x_1 - k_2x_1 + k_2x_2 = -(k_1 + k_2)x_1 + k_2x_2$$

- Block 2 (mass m_2): When $x_1(t)$ and $x_2(t)$ are both to the right of the equilibrium position, a force $-k_3x_2$ acts to oppose the displacement of the block from the right. Similarly, the spring k_2 opposes the motion of the block from the left giving component $-k_2x_2$. However, since block 1 (mass m_1) is also displaced to the right by amount x_1 this causes spring k_2 to compress and push m_2 to the right with force k_2x_1 (proportional to how much m_1 is moved - in opposing the motion of m_1 the spring exerts an equal forced push on m_2). Hence, the net force on block m_2 is:
$$F_2 = m_2\frac{d^2x_2}{dt^2} = -k_3x_2 - k_2x_2 + k_2x_1 = k_2x_1 - (k_2 + k_3)x_2$$

Notice that we can write these two differential equations for x_1'' and x_2'' as:

$$\begin{aligned}
\frac{d^2x_1}{dt^2} &= \frac{-(k_1+k_2)}{m}x_1 + \frac{k_2}{m}x_2 = ax_1 + bx_2 \\
\frac{d^2x_2}{dt^2} &= \frac{k_2}{m}x_1 + \frac{-(k_2+k_2)}{m}x_2 = cx_1 + dx_2
\end{aligned}$$

If we now introduce two new variables $x_3 = x_1'$ and $x_4 = x_2'$ we can rewrite the equations above as:

$$\begin{aligned}
x_1'' = x_3' = ax_1 + bx_2 \\
x_2'' = x_4' = cx_1 + dx_2
\end{aligned}$$

so that the full system of first order equations becomes:

$$\begin{aligned}
x_1' &= x_3 \\
x_2' &= x_4 \\
x_3' &= ax_1 + bx_2 \\
x_4' &= cx_1 + dx_2
\end{aligned}$$

We can rewrite this in matrix form as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & 0 & 0 \\ c & d & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Leftrightarrow \vec{x}'(t) = A\vec{x}(t) = A \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

Many other physical systems can be modeled by linear differential equation systems $\vec{x}'(t) = A\vec{x}(t)$ with A an $n \times n$ matrix. Note once again, that higher order systems can be reduced to first order systems via substitution, as we did above.

To find solutions to $\vec{x}'(t) = A\vec{x}(t)$, we try the substitution $\vec{x} = e^{\lambda t}\vec{v} \implies \vec{x}' = \lambda e^{\lambda t}\vec{v}$ to get: $\lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v}$. Rearranging we get:

$$e^{\lambda t}(A - \lambda I)\vec{v} = \vec{0} \implies (A - \lambda I)\vec{v} = \vec{0}$$

since $e^{\lambda t} > 0$. Thus, we have that $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution to the system $\vec{x'}(t) = A\vec{x}(t)$ when λ is an eigenvalue and $\vec{v}$ is a corresponding eigenvector of matrix A. At this point there are several cases, depending on the eigenvalue and eigenvectors which A has. We will restrict our analysis to 2×2 systems, that is $A \in \mathbb{R}^{2\times 2}$. The solution to the system depends on the eigenvalues and eigenvectors of A.

- (1) A has n distinct real eigenvalues and n linearly independent eigenvectors. Then for each pair $\lambda_i, \vec{v_i}$ it follows that $\vec{x_i}(t) = e^{\lambda_i t}\vec{v_i}$ is a solution to the system and the general solution is a general linear combination of all n solution vectors. In the 2×2 case, the solution is:

$$\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v_1} + C_2 e^{\lambda_2 t}\vec{v_2}$$

  If both $\lambda_1, \lambda_2 < 0$, then all trajectories go back to the equilibrium point $(0,0)$ as $t \to \infty$. This results in a stable node. The non-straight line trajectories will be parabolas (see derivation later in this pdf). If the eigenvalues are of different signs, the result will be a saddle point for $(0,0)$, some trajectories will go to $(0,0)$ and others away from it. The non-straight line trajectories will be hyperbolas in this case. If the eigenvalues are both positive, all trajectories will go away from $(0,0)$ with increasing t and an unstable node will result and non-straight line trajectories will be parabolas.

- (2) A has complex conjugate eigenvalues and complex eigenvectors. Note that if A is say 2×2 then if one eigenvalue is complex, the other eigenvalue is just a complex conjuate of that eigenvalue and the same relationship holds for the eigenvectors. That is, if the first eigenvalue is say $\lambda_1 = \alpha + i\beta$ then the second eigenvalue must be $\lambda_2 = \alpha - i\beta$ and if the first eigenvector is $\vec{v}_1$ then the second eigenvector is the complex conjugate of $\vec{v}_1$ (replace all i by $-i$). However for a 3×3 matrix, we can have for example one real eigenvalue and two complex conjugate eigenvalues, but the complex ones always occur in conjugate pairs. Now consider a complex eigenvector pair $\vec{v}_1, \vec{v}_2 = \vec{p} \pm i\vec{q}$ corresponding to two complex eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$. Let us take one of the eigenvalues and eigenvectors λ_1 and $\vec{v}_1$ and form a complex valued solution. Then:

$$\begin{aligned}
\vec{x}_{cmp}(t) = e^{\lambda_1 t}\vec{v}_1 &= e^{(\alpha+i\beta)t}\left(\vec{p} + i\vec{q}\right) = e^{\alpha t}e^{i\beta t}\left(\vec{p} + i\vec{q}\right) = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)\left(\vec{p} + i\vec{q}\right) \\
&= e^{\alpha t}\left(\cos(\beta t)\vec{p} - \sin(\beta t)\vec{q}\right) + ie^{\alpha t}\left(\sin(\beta t)\vec{p} + \cos(\beta t)\vec{q}\right)
\end{aligned}$$

  Note that both the real ($Re(\vec{x}_{cmp}(t))$) and imaginary ($Im(\vec{x}_{cmp}(t))$) parts of the complex valued solution vector $\vec{x}_{cmp}(t)$ are real functions. That is $\vec{x}_{re}(t) = e^{\alpha t}\left(\cos(\beta t)\vec{p} - \sin(\beta t)\vec{q}\right)$ and $\vec{x}_{im}(t) = e^{\alpha t}\left(\sin(\beta t)\vec{p} + \cos(\beta t)\vec{q}\right)$ are real and satisfy:

$$\vec{x}'(t) = \vec{x}'_{re}(t) + i\vec{x}'_{im}(t) = A\vec{x}(t) = A\vec{x}_{re}(t) + iA\vec{x}_{im}(t)$$

  Hence, equating the real and imaginary parts, we have that:

$$\vec{x}'_{re}(t) = A\vec{x}_{re}(t) \quad \text{and} \quad \vec{x}'_{im}(t) = A\vec{x}_{im}(t)$$

  So that $\vec{x}'_{re}(t)$ and $\vec{x}'_{im}(t)$ are two real solutions to the linear ODE system corresponding to complex conjugate eigenvalues/eigenvectors $\lambda_{1,2}$, $\vec{v}_{1,2}$. Notice again that it doesn't matter which eigenvalue and eigenvector pair we use to form the complex valued solution.

Example: Let:

$$\vec{x}'(t) = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \vec{x}(t)$$

One then finds that the eigenvalues are $\lambda_{1,2} = -\frac{1}{2} \pm i$ and the eigenvectors are $\vec{v}_{1,2} = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$. We take any one of the eigenvalues and eigenvectors and extract the real and imaginary parts as two real solutions from the complex valued solution vector:

$$\vec{x}(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{\left(-\frac{1}{2}+i\right)t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{t}{2}} \left(\cos(t) + i\sin(t)\right) = \begin{bmatrix} e^{-\frac{t}{2}}\cos(t) \\ -e^{-\frac{t}{2}}\sin(t) \end{bmatrix} + i \begin{bmatrix} e^{-\frac{t}{2}}\sin(t) \\ e^{-\frac{t}{2}}\cos(t) \end{bmatrix}$$

That is, the two solutions are $\vec{x}_{re}(t) = \begin{bmatrix} e^{-\frac{t}{2}}\cos(t) \\ -e^{-\frac{t}{2}}\sin(t) \end{bmatrix}$ and $\vec{x}_{im}(t) = \begin{bmatrix} e^{-\frac{t}{2}}\sin(t) \\ e^{-\frac{t}{2}}\cos(t) \end{bmatrix}$ and the general solution is:

$$\vec{x}(t) = C_1 e^{-\frac{t}{2}} \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2 e^{-\frac{t}{2}} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

- (3) A has repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$ but two linearly dependent vectors for the repeated eigenvalue. In this case we have eigenvectors $\vec{v}_1$ and $\vec{v}_2$ and the general solution is simply:

$$\vec{x}(t) = C_1 e^{\lambda t}\vec{v}_1 + C_2 e^{\lambda t}\vec{v}_2$$

  In this case, $\frac{x(t)}{y(t)} = K$ and the solution trajectories correspond to a star node.

- (4) A has repeated eigenvalues and for one or more of the repeated eigenvalues, there is less linearly independent eigenvectors than the number of times the eigenvalue is repeated. Suppose we have a 2×2 matrix with one real repeated eigenvalue λ but only one linearly independent eigenvector $\vec{v}$. Then one solution is $\vec{x}_1(t) = e^{\lambda t}\vec{v}$. If we try for another solution $\vec{x}(t) = te^{\lambda t}\vec{v}$ (motivated by second order constant coefficient equations case with a double root of the characteristic equation), then for this vector to satisfy the ODE system, we have:

$$\vec{x}'(t) = e^{\lambda t}\vec{v} + \lambda t e^{\lambda t}\vec{v} = A(te^{\lambda t}\vec{v})$$

  which is satisfied only for $\vec{v} = 0$. This, however, is inconsistent with $\vec{v}$ being an eigenvector. It turn out that if we take as the second solution $\vec{x}_2(t) = e^{\lambda t}(t\vec{v} + \vec{u})$ this works provided $\vec{u}$ satisfies the linear system $(A - \lambda I)\vec{u} = \vec{v}$. To see this, simply pluging $\vec{x}_2(t)$ into the ODE system $\vec{x}_2'(t) = A\vec{x}_2(t)$ we get:

$$\begin{aligned} &\frac{d}{dt}\left(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{u}\right) = A\left(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{u}\right) \implies e^{\lambda t}\vec{v} + \lambda t e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{u} = (A\vec{v})te^{\lambda t} + (A\vec{u})e^{\lambda t} \\ &\implies (\vec{v} + \lambda\vec{u})\, e^{\lambda t} + (\lambda\vec{v})te^{\lambda t} = (A\vec{u})e^{\lambda t} + (A\vec{v})te^{\lambda t} \end{aligned}$$

  This imples that $\vec{x}_2$ is a solution if the following hold:

$$A\vec{v} = \lambda\vec{v} \quad \text{and} \quad \vec{v} + \lambda\vec{u} = A\vec{u} \implies (A - \lambda I)\vec{u} = \vec{v}$$

The first condition is simply the eigenvector condition on $\vec{v}$ and the second condition is $(A - \lambda I)\vec{u} = \vec{v}$ which $\vec{u}$ has to satisfy. Note that $\vec{u}$ is known as a generalized eigenvector. That's because $(A - \lambda I)^2\vec{u} = (A - \lambda I)\vec{v} = 0$. The general solution is then given by:

$$\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) = C_1e^{\lambda t}\vec{v} + C_2e^{\lambda t}(t\vec{v} + \vec{u})$$

See the handwritten notes at the end of the pdf for examples of this case.

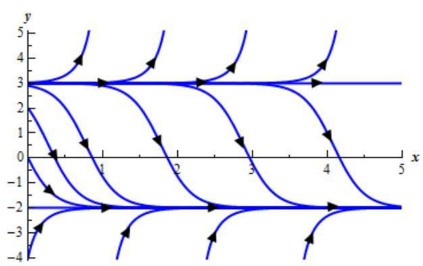
References:

- Paul's notes: http://tutorial.math.lamar.edu/Classes/DE/SystemsIntro.aspx
- Diffeq for Engineers book (chapter 3): http://www.jirka.org/diffyqs/

3 Stability of linear ODE systems (Textbook: section 6.4)

As an illustration condsider the single ODE $x'(t) = \alpha x(t)$ which models exponential growth or decay for $t \geq 0$. The solution, by separation of variables, is $x(t) = Ce^{\alpha t}$. An equilibrium solution to the equation is $x(t) = 0$ corresponding to the initial condition $x(0) = 0$. That is, if at time $t = 0$, we have $x(0) = 0$ and follow the above model, the $x(t)$ will remain zero for all time t. However, if we perturb the initial condition slightly, starting off at say $x(0) = 0.001$, the long term behavior of the solution will depend on α. For $\alpha > 0$, $x(t) \to \infty$ as $t \to \infty$ while for $\alpha < 0$, $x(t) \to 0$ as $t \to \infty$. We say that for $\alpha < 0$, the zero equilibrium solution $x(t) = 0$ is asymptotically stable (because solutions which start near $x(0) = 0$ will remain nearby as $t \to \infty$. On the other hand, for $\alpha > 0$, the solution $x(t) = 0$ is asymptotically unstable. If we start just a bit off $x(0) = 0$ (say at $x(0) = 0.001$), then we will go very far away from $x(t) = 0$ as t gets large.

For an equation such as $y'(t) = y^2 - y - 6$, there are multiple equilibrium solutions given by $y^2 - y - 6 = (y - 3)(y + 2) = 0 \implies y = 3, y = -2$. A sketch of the integral curves (basically, a few solution curves corresponding to different initial conditions of the ODE) gives the following plot: This can be sketeched by hand, simply by considering the signs of the tangent vectors to the solution



of the ODE (that is, the values of $y'(t)$ given by $(y - 3)(y + 2)$ for different values of y). Notice that since the ODE is autonomous (no dependence on t), the slope values depend just on y.

We can talk also about stability of equilibrium solutions of linear ODE systems. Consider the 2×2 constant coefficient linear ODE system:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

which we write compactly as $\vec{x}'(t) = A\vec{x}(t)$. Clearly, $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ when $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so we call $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ an equilibrium solution (or equilibrium point) of the ODE system. We consider the initial condition (at $t = 0$), $\vec{x}(0) = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$ for small ϵ_1, ϵ_2 and want to know what happens to the vector solution of the ODE as $t \to \infty$ (asymptotic stability analysis). That is, we would like to classify the equilibrium point of the ODE system, as we did with single differential equations above. The behavior clearly depends on the solution of the system, which is governed by it's eigenvalues and eigenvectors. For 2×2 systems the eigenvalues are given by $|A - \lambda I| = 0 = (a - \lambda)(d - \lambda) - bc = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = 0$. Note that the sum of the diagonal elements $a + d$ is the trace of the matrix $Tr(A)$ and $ad - bc = |A|$. Hence, the eigenvalues of the matrix A can be expressed in terms of it's trace and determinant. The formula below is valid only for the 2×2 case:

$$\lambda^2 - Tr(A) + |A| = 0 \implies \lambda_{1,2} = \frac{Tr(A) \pm \sqrt{Tr^2(A) - 4|A|}}{2}$$

Based on the eigenvalues λ_1, λ_2 we can conclude the following (see handwritten notes at end for more detailed analysis):

- Real and distinct positive eigenvalues, $\lambda_1 \neq \lambda_2$, $\lambda_1 > 0, \lambda_2 > 0$: Both solutions have a positive exponential term $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ which go to ∞ as $t \to \infty$. Hence, this situation corresponds to an unstable source. Solution trajectories starting near the origin, go futher and further away from the origin as time increased.

- Real and distinct negative eigenvalues $\lambda_1 \neq \lambda_2$, $\lambda_1 < 0, \lambda_2 < 0$. This is the opposite of the previous case and the trajectories form a stable sink. Solution trajectories starting near the origin stay near the origin and eventually go to the origin as $t \to \infty$.

- Real eigenvalues of different signs, $\lambda_1 \neq \lambda_2$, $\lambda_1 < 0, \lambda_2 > 0$. If the eigenvalues have opposite signs, some solutions will approach the origin and others will go away from the origin as $t \to \infty$. The trajectories form an unstable saddle.

- Complex eigenvalues $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$. These always occur in complex conjugate pairs. The behavior depends on the sign of α as that corresponds to the term $e^{\alpha t}$ in the solution. If $\alpha = 0$, then the trajectories will be circular around the origin with constant radius for all t. These are referred to as stable centers. On the other hand if $\alpha < 0$, the trajectories are stable spirals and if $\alpha > 0$ they are unstable spirals, whose radius increases with time.

- Repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$. Here the situation depends on the number of linearly independent eigenvectors associated with λ. There can be either one or two. For the 2×2 case, 2 is only possible if $A = \lambda I$, that is A is a multiple of the diagonal identity matrix. In that case $(A - \lambda I) = 0$ and any vector in $\mathbb{R}^2$ is an eigenvector, since for any $\vec{v} \in \mathbb{R}^2$, $(A - \lambda I)\vec{v} = 0$. Two linearly independent eigenvectors are then the standard $\vec{e}_1$ and $\vec{e}_2$. The trajectories form star nodes (straight lines through zero) which either go inward or outward depending on sign of λ. They are then stable or unstable star nodes. If there is only a single linearly independent eigenvector, then the solution is:

$$\vec{x}(t) = C_1 e^{\lambda t}\vec{v} + C_2 e^{\lambda t}(t\vec{v} + \vec{u})$$

The trajectories are so called degenerate nodes which can be either stable or unstable depending on the sign of λ (stable when $\lambda < 0$). Note that as $|e^{\lambda t} t| > |e^{\lambda t}|$ for large t, the solution eventually becomes almost parallel to the vector $\vec{v}$. The star and degenerate nodes occur on the curve of the parabola on the stability curve (illustrated below) as the case of equal eigenvalues corresponds to $\sqrt{Tr^2(A) - 4|A|} = 0$.

Notice that the different types of trajectories can be identified based on the values of $Tr(A)$ and $|A|$ (for 2×2 systems). The figure below shows trajectory types as a function of the two. Notice that on the curve $Tr^2(A) = 4|A|$ which is a parabola, we have degenerate or star nodes since this is the case corresponding to equal eigenvalues and we must check how many linearly independent eigenvectors there are to determine the type of node.

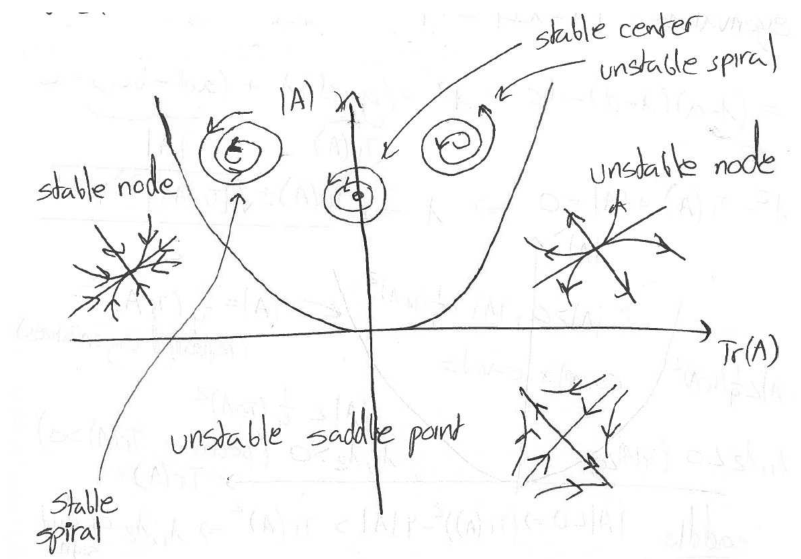


Figure 1: Stability diagram for 2×2 linear constant coefficient systems relating trajectory types to eigenvalues depending on values of $Tr(A)$ and A.

4 Nonlinear Sytems of ODEs and Linearization (Textbook: Section 7.1 - 7.2)

Nonlinear equations and systems are much harder to analyze then linear ones and the behavior can be a lot more complicated. Consider the autonomous system (right hand side does not explicitly contain the independent vaiable):

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}$$

where f and g are nonlinear functions. Since it's nonlinear, this system cannot be written in the form $\vec{x}' = A\vec{x}$. We have two fairly simple tools in our disposal for analyzing the behavior around equlibrium points (where x' and y' are both zero). One is the method of nullclines (see handwritten notes) and the other is linearization around the equilibrium points by means of taking only the linear terms of the Taylor expansions. We now discuss the second approach. Around each equilibrium point (x_e, y_e) (i.e. where $f(x_e, y_e) = 0 = g(x_e, y_e)$) we write a linearized system using the substitutions $u = x - x_e$ and $v = y - y_e$, where x, y, u, v are functions of t. Keeping only the first order terms of the Taylor expansions of $f(u, v)$ around (x_e, y_e), this results in the linear system:

$$\begin{bmatrix} u \\ v \end{bmatrix}' = J(x_e, y_e) \begin{bmatrix} u \\ v \end{bmatrix} \tag{4.1}$$

where the Jacobian matrix of the system above is given by:

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \implies J(x_e, y_e) = \begin{bmatrix} f_x(x_e, y_e) & f_y(x_e, y_e) \\ g_x(x_e, y_e) & g_y(x_e, y_e) \end{bmatrix} \tag{4.2}$$

Now, if the Jacobian matrix $J(x_e, y_e)$ in (4.1) is nonsingular (that is, it's determinant $|J(x_e, y_e)| \neq 0$), then there is a unique equilibrium point of this system $(0, 0)$ and the behavior around (x_e, y_e) of the nonlinear system can be found (in some cases) by finding the stability properties of the linearized system around $(0, 0)$. If $J(x_e, y_e)$ is singular (zero determinant) no conclusion can be reached with this method about stability properties of the nonlinear system around (x_e, y_e). Also, if $J(x_e, y_e)$ has purely imaginary eigenvalues so that around $(0, 0)$ of the linearized system there are circles, the behavior around (x_e, y_e) of the nonlinear system can be said to be either that of circles, unstable or stable spirals (that is, we cannot conclude stability in this case). In all other cases, we expect the behavior of the nonlinear system around (x_e, y_e) to match the behavior of the linearized system around (x_0, y_0).

Ex1. Consider the equation:

$$\frac{d^2x}{dt^2} - (1 - x^2)\frac{dx}{dt} + x = 0$$

Let $y(x) = x'(t)$. Then $y' = x'' = (1 - x^2)\frac{dx}{dt} - x = (1 - x^2)y - x$. Thus, we get the equivalent system:

$$\begin{aligned} x' &= y \\ y' &= (1 - x^2)y - x \end{aligned}$$

The only equilibrium point is $(0,0)$, when $x=0, y=0$. Here, $f(x,y)=y$ and $g(x,y)=(1-x^2)y-x=-x+y-x^2y$. The Jacobian is:

$$J=\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -1-2xy & 1-x^2 \end{bmatrix} \implies J(0,0)=\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \tag{4.3}$$

The linearized system around $(0,0)$ is thus given by:

$$\begin{bmatrix} u \\ v \end{bmatrix}'=\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \tag{4.4}$$

Since $u=x-x_e=x-0=x$ and $v=y-y_e=y-0=y$, we get the linearized system:

$$\begin{aligned} x' &= y \\ y' &= -x+y \end{aligned}$$

Since we have $|J|=1$ and $Tr(J)=1$, the eigenvalues are given by:

$$\lambda_{1,2}=\frac{Tr(J)\pm\sqrt{Tr^2(J)-4|J|}}{2}=\frac{1\pm i\sqrt{3}}{2}$$

Since the real part of the eigenvalues is positive, the solutions of the linearized system spiral away from the origin (unstable spiral) and we expect the solutions of the nonlinear system to spiral away from the origin as well.

Reference: http://www.sosmath.com/diffeq/system/nonlinear/linearization/linearization.html.

Phase plots and general solutions of 2x2 ODE systems (linear, constant coefficient)

We are interested, in this case, in systems written in the form:

$$\vec{x}'(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}(t) = A\vec{x}(t)$$

with $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \Rightarrow A\vec{x}(t) = \begin{bmatrix} ax(t) + by(t) \\ cx(t) + dy(t) \end{bmatrix}$

So that $\vec{x}'(t) = A\vec{x}(t)$ is equivalent to the two equations

$$x'(t) = ax(t) + by(t)$$
$$y'(t) = cx(t) + dy(t)$$

$\left(x'(t) = \frac{dx}{dt} \; ; \; y'(t) = \frac{dy}{dt}\right)$

If x' depends only on x and y' only on y then the system is "decoupled" meaning that the two equations can be solved independently. Otherwise, the system is coupled and at least one of ~~each~~ ~~of~~ x', y' depend on x, y.

The phase plot is a set of solution curves $\langle x(t), y(t)\rangle = x(t)\hat{i} + y(t)\hat{j}$ in the x-y plane (each corresponding to some initial set of conditions

(1) A has real eigenvalues of <u>same</u> sign.

$\lambda_1, \lambda_2 > 0$ or $\lambda_1, \lambda_2 < 0$. There are 2 linearly indep. eigenvectors $\vec{v}_1, \vec{v}_2 \Rightarrow \vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$

Ex] $\begin{matrix} x' = -x \\ y' = -2y \end{matrix} \iff \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \iff \vec{x}' = A\vec{x}$

with $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = (-1-\lambda)(-2-\lambda) = 0$

$\Rightarrow \lambda_1 = -1 \; ; \; \lambda_2 = -2$

$(A - \lambda_1 I)\vec{v}_{(1)} = \vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_2 = 0$; v_1 can be any $\alpha \in \mathbb{R}$ $\Rightarrow \vec{v}_{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (or any multiple of this)

Similary, seecond eigenvector is $\vec{v}_{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$ is general solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{-2y}{-x} = \frac{2y}{x}$$

$$\Rightarrow \frac{dy}{y} = 2\frac{dx}{x} \Rightarrow \ln|y| = 2\ln|x| + c_1$$

$$\Rightarrow |y| = e^{2\ln|x|} e^{c_1} \Rightarrow y = ce^{2\ln|x|} = ce^{\ln x^2} = cx^2$$

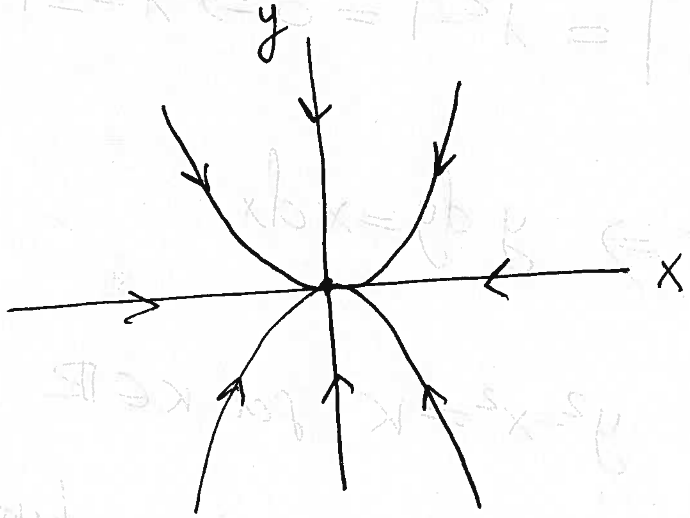
So the solution curves satisfy $y = Cx^2$ and are parabolas. In general, we will have $y = Cx^n$ for some n, depending on β, Γ in $x' = \beta x$ and $y' = \Gamma y$

"Eigensolutions" : special solutions from general solution

$$\pm\begin{pmatrix}1\\0\end{pmatrix}e^{-t} \text{ and } \pm\begin{pmatrix}0\\1\end{pmatrix}e^{-2t}$$

These travel along x-axis and y-axis respectively.

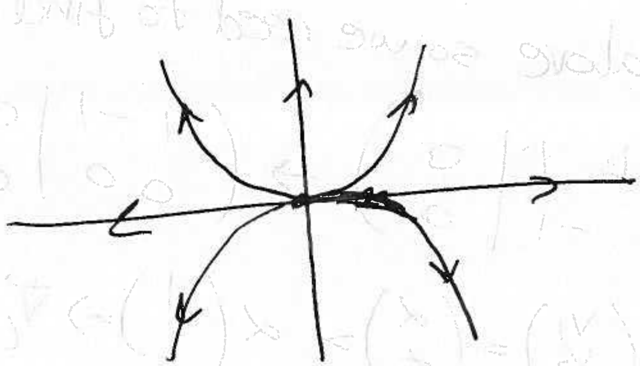
The whole phase plot corresponds to a sink node.



stable sink node

everything around $\begin{pmatrix}0\\0\end{pmatrix}$ equilibrium point tends towards it. That's because $\lambda_1, \lambda_2 < 0$.

In the case when $\lambda_1, \lambda_2 > 0$ we get:



unstable source node

(2) A has real eigenvalues of different signs ($\lambda_1 > 0, \lambda_2 < 0$ or $\lambda_1 < 0, \lambda_2 > 0$).

In this case we still have 2 linearly indep eigenvectors $\vec{v}_1, \vec{v}_2 \Rightarrow \vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$ is the general solution.

Ex] $\vec{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x} \Leftrightarrow \begin{array}{l} x' = y \\ y' = x \end{array}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x}{y} \Rightarrow y\,dy = x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + K_1 \Rightarrow y^2 - x^2 = K \text{ for } K \in \mathbb{R}$$

K can be >0, or <0, these are a family of hyperbolas facing all 4 directions in x-y plane.

The general solution is as above so we need to find $\vec{v}_1, \vec{v}_2$.

$$(A - \lambda I)\vec{v}_{(1)} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let

$$\Rightarrow v_1 - v_2 = 0 \Rightarrow \vec{v}_{(1)} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{v}_{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Similarly, $(A-\lambda_2 I)\vec{v}_{(2)} = \begin{pmatrix}0\\0\end{pmatrix} = \left(\begin{array}{cc|c}1&1&0\\1&1&0\end{array}\right) \to \left(\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right)$

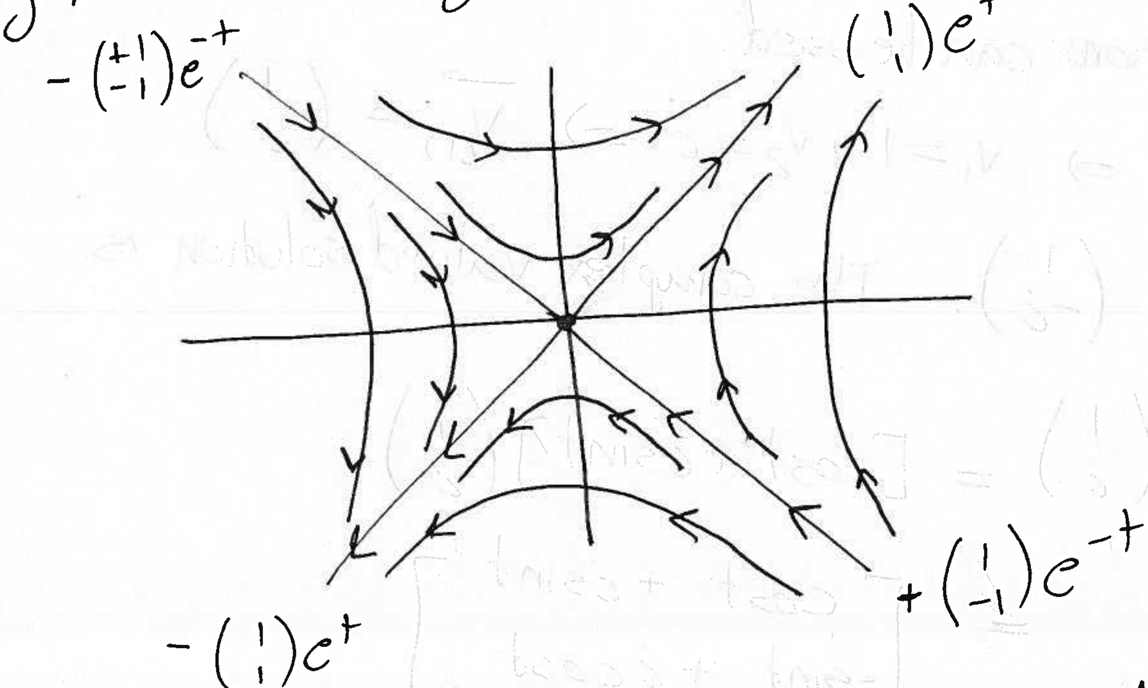
$v_1+v_2=0 \Rightarrow -v_1=+v_2 \quad \Rightarrow \quad \vec{v}_{(2)} = \beta\begin{pmatrix}1\\-1\end{pmatrix}$

$v_2=\beta$

Let $\vec{v}_{(2)} = \begin{pmatrix}1\\-1\end{pmatrix}$. Thus, solution is:

$$\vec{x}(t) = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}$$

eigensolutions: $\pm\begin{pmatrix}1\\1\end{pmatrix}e^{t}$ and $\pm\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}$ lie along lines $y=x$ and $y=-x$. The solution along $y=-x$ travels towards origin since $\lambda_2=-1<0$. The solution along $y=x$ travels away from origin. The phase plot looks like this.



unstable

saddle

along some directions solutions travel towards origin, along others they go away from origin.

(3) A has completely complex eigenvalues $\lambda = \pm \beta i$ with no real part. (i.e. $\lambda = \alpha \pm i\beta$ with $\alpha = 0$).

Ex) $\vec{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x} \iff \begin{matrix} x' = y \\ y' = -x \end{matrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$(A - \lambda_1 I) = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since $\operatorname{rank}(A - \lambda_1 I) = 1$ (b/c $\det(A - \lambda_1 I) = 0$) any of the two equations can be used.

$-iv_1 + v_2 = 0 \Rightarrow v_1 = 1;\ v_2 = i \Rightarrow \vec{v}_{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Thus, $\vec{v}_{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. The complex valued solution is

$$\vec{x}_{comp}(t) = e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} = [\cos t + i \sin t] \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{bmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{bmatrix}$$

Taking real and imaginary parts of $\vec{x}_{comp}(t)$ we get:

$$\vec{x}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-x}{y} \Rightarrow -x\,dx = y\,dy$$

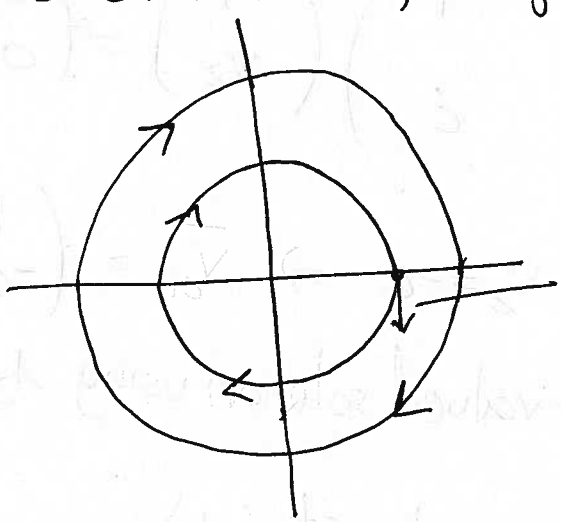
$$\Rightarrow \quad -\frac{x^2}{2} + K_1 = \frac{y^2}{2} \Rightarrow x^2 + y^2 = K$$

(this is a family of concentric circles). To determine orientation, ~~plea~~ compute tangent vector, say at $\binom{1}{0}$:

since $\vec{x}\,'(t) = A\vec{x}(t)$ then $\vec{x}\,'\big|_{\binom{1}{0}} = A\binom{1}{0} =$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\hat{j}$$. This means

motion is clockwise, see phase plot below:



tangent vector to $\binom{1}{0}$ is $\binom{0}{-1}$

concentric circles

stable center

(simply called "center")

(4) A has complex eigenvalues with nonzero real part

$\lambda = \alpha \pm i\beta$ with $\alpha \neq 0$

Ex]

$$x' = -x + y \qquad \Longleftrightarrow \qquad \vec{x}'(t) = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$
$$y' = -x - y$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 + 1 =$$

$$= (\lambda+1)^2 + 1 = \lambda^2 + 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Take $\lambda_1 = -1 - i$; $\lambda_2 = -1 + i$

$$\cancel{\det}(A - \lambda_1 I)\vec{V}_{(1)} = \vec{0} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow iV_1 + V_2 = 0 \Rightarrow V_1 = 1;\ V_2 = -i \Rightarrow \vec{V}_{(1)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\vec{V}_{(2)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Write complex valued solution using $\lambda_2, \vec{V}_{(2)}$:

$$\vec{x}_{comp}(t) = e^{(-1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-t} e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} =$$

$$= e^{-t}(\cos t + i \sin t) \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-t} \begin{pmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{pmatrix}$$

$$\vec{X}_1(t) = Re\left[\vec{X}_{comp}(t)\right] = e^{-t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$\vec{X}_2(t) = Im\left[\vec{X}_{comp}(t)\right] = e^{-t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\Rightarrow \vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

$$= c_1 e^{-t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \Rightarrow \begin{array}{l} x(t) = e^{-t}(c_1 \cos t + c_2 \sin t) \\ y(t) = e^{-t}(-c_1 \sin t + c_2 \cos t) \end{array}$$

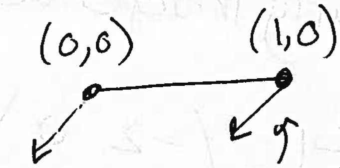
$$x^2+y^2 = e^{-2t}\left[c_1^2\cos^2 t + c_2^2 \sin^2 t + 2c_1c_2 \cos t \sin t + c_1^2 \sin^2 t + c_2^2\cos^2 t - 2c_1c_2\cos t \sin t\right]$$

$$= e^{-2t}\left[c_1^2 + c_2^2\right] = Ke^{-2t}$$

$\Rightarrow x^2+y^2 = e^{-2t}$ represents a shrinking spiral

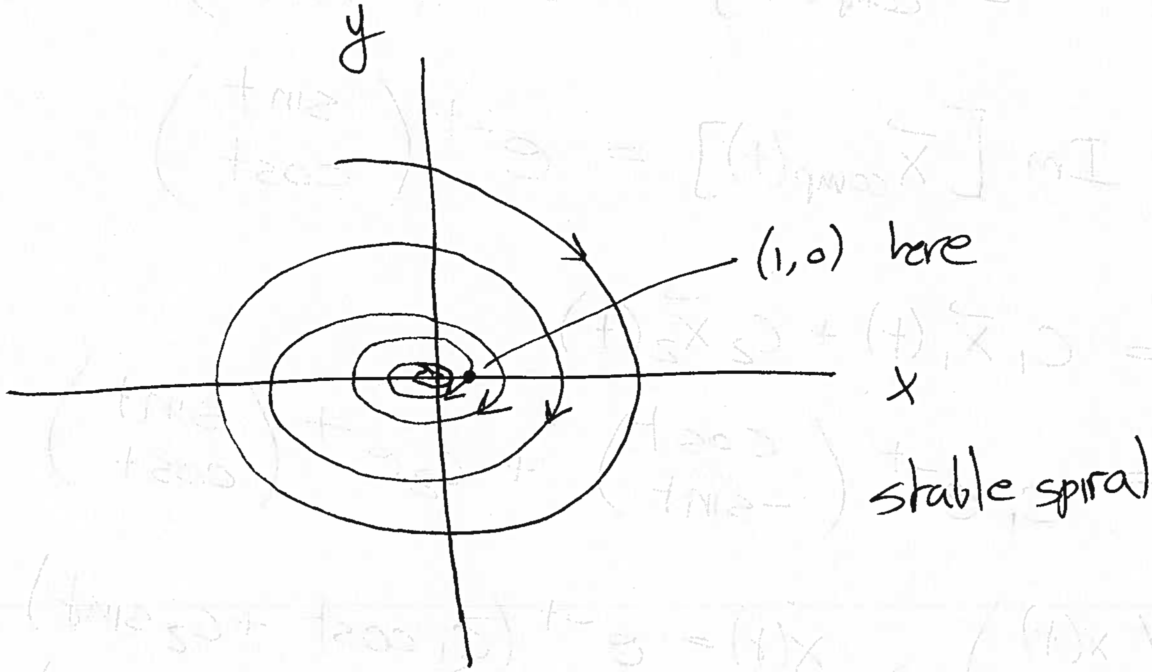
To draw properly, check tangent vector say at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\vec{x}'\Big|_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$



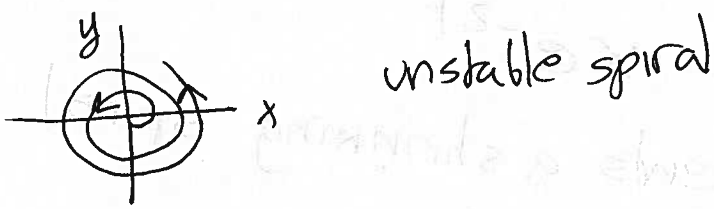
tangent vector at (1,0) points here

Thus, the spiral would look like



The spiral is stable because $Re(-1 \pm i) = -1 < 0$.
All solutions tend back to equilibrium point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.

If we had $\lambda = \alpha \pm i\beta$ with $\alpha > 0$ the spiral would spin outward (be unstable). For example.



(5) A has repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$ with two linearly independent eigenvectors $\vec{V}_{(1)}, \vec{V}_{(2)}$.

$\underline{\text{Ex}}$ $\vec{X}' = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \vec{X} \iff \left.\begin{aligned} dx/dt &= -2x \\ dy/dt &= -2y \end{aligned}\right\}$ decoupled system

$\lambda_1 = \lambda_2 = -2$ $\quad A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$\Rightarrow (A - \lambda I)\vec{v} = \vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_1, v_2 \in \mathbb{R}$ (any choice works). Any vector $\vec{v} \in \mathbb{R}^2$ is an eigenvector so two lin indep vecs are:

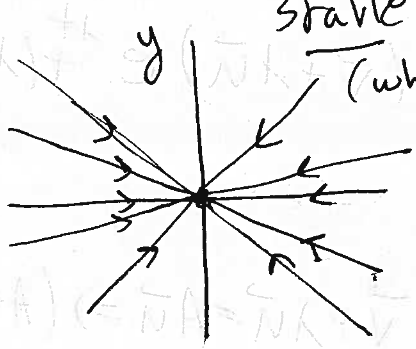
$\vec{v}_{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

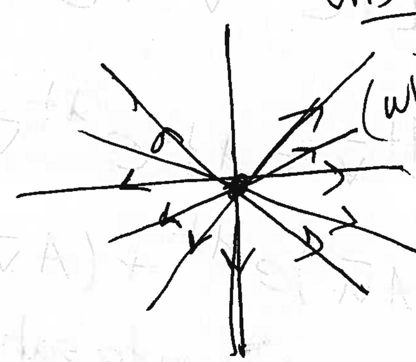
$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 \vec{v}_2 e^{\lambda t} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$$

eigensolutions: $\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}$; $\pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$ travel along x, y axis.

$$\frac{x(t)}{y(t)} = \frac{c_1 e^{-2t}}{c_2 e^{-2t}} = \frac{c_1}{c_2} = K \Rightarrow x = Ky$$

These are straight lines ~~for the~~ through the origin. We get so called star nodes for the phase plot.





(6) A in $\vec{x}' = A\vec{x}$ has repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$ with only one linearly independent eigenvector $\vec{v}_{(1)}$.

Ex

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \vec{x} \Rightarrow \lambda_1 = \lambda_2 = -2 = \lambda$$

$$(A - \lambda I)\vec{v} = \vec{0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_2 = 0, \; v_1 = \alpha \in \mathbb{R} \Rightarrow \vec{v}_{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

General solution given by:

$$\vec{x} = c_1 e^{\lambda t} \vec{v} + c_2 \left[t e^{\lambda t} \vec{v} + e^{\lambda t} \vec{u} \right]$$

where $\vec{u}$ is a generalized eigenvector, see below.

For $t e^{\lambda t} \vec{v} + e^{\lambda t} \vec{u}$ to be a solution of $\vec{x}' = A\vec{x}$ we must have:

$$\frac{d}{dt}\left[t e^{\lambda t} \vec{v} + e^{\lambda t} \vec{u} \right] = A \left[t e^{\lambda t} \vec{v} + e^{\lambda t} \vec{u} \right]$$

$$\Rightarrow e^{\lambda t} \vec{v} + \lambda t e^{\lambda t} \vec{v} + \lambda e^{\lambda t} \vec{u} = (\vec{v} + \lambda \vec{u}) e^{\lambda t} + (\lambda \vec{v}) t e^{\lambda t}$$

$$= (A\vec{u}) e^{\lambda t} + (A\vec{v}) t e^{\lambda t}$$

$$\Rightarrow A\vec{v} = \lambda \vec{v} \left(\begin{matrix} \text{auto satisfied} \\ \text{since } \vec{v} \text{ eigenvector} \end{matrix} \right) \text{ and } \vec{v} + \lambda \vec{u} = A\vec{u} \Rightarrow (A - \lambda I)\vec{u} = \vec{v}$$

Notice that $(A-\lambda I)\vec{u}=\vec{v}$ implies:

$$(A-\lambda I)(A-\lambda I)\vec{u} = (A-\lambda I)\vec{v}=\vec{0}$$

since $A\vec{v}=\lambda\vec{v} \Longleftrightarrow (A-\lambda I)\vec{v}=\vec{0}$

$\Rightarrow (A-\lambda I)^2\vec{u}=\vec{0}$ ($\vec{u}$ is called generalized eigenvector).

In this case,

$$(A-\lambda I)\vec{u}=\vec{v} \Longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

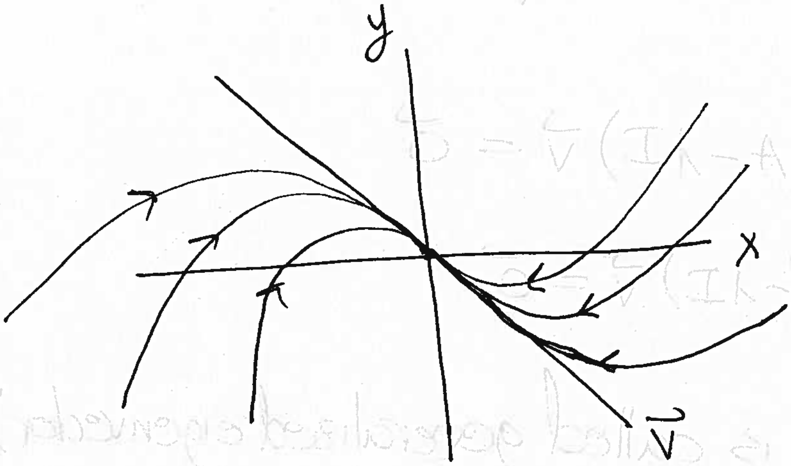
$\Rightarrow u_1=\alpha\in\mathbb{R}$; $u_2=1$

There are infinitely many choices for $\vec{u}$. We can take $u_1=0$. Then $\vec{u}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and general solution is:

$$\vec{x}(t)=c_1e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}+c_2\left[te^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}+e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

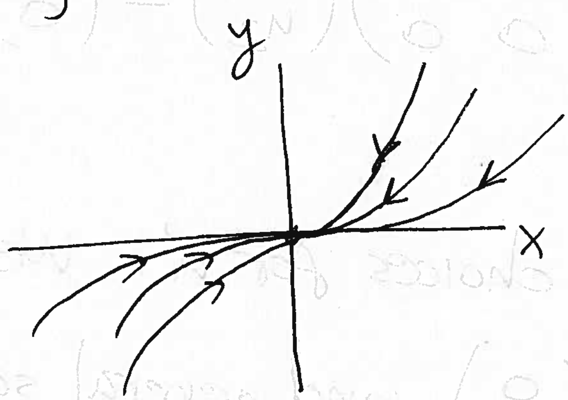
As $t\to\infty$, te^{-2t} dominates since $|te^{-2t}|>|e^{-2t}|$ for large t. Thus, trajectories become parallel to the eigensolution $e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $t\to\pm\infty$. That is, they become almost parallel to eigenvector $\vec{v}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for large $|t|$.

In general the degenerate node looks like this:



stable degenerate node (when $\lambda < 0$)

In our case $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so the phase plot looks something like this:



since $\vec{v}$ coincides with x-axis.

when $\lambda > 0$, we ~~we~~ would have an unstable degenerate node pointing outward.

Ex) $\vec{x}' = \begin{pmatrix} 12 & 4 \\ -16 & -4 \end{pmatrix} \vec{x}$

$$|A - \lambda I| = \begin{vmatrix} 12-\lambda & 4 \\ -16 & -4-\lambda \end{vmatrix} = (12-\lambda)(-4-\lambda) + 64$$

$$= -48 - 12\lambda + 4\lambda + \lambda^2 + 64 = \lambda^2 - 8\lambda + 16 = 0$$

$$= (\lambda - 4)^2 \Rightarrow \lambda_1 = \lambda_2 = 4$$

$$(A - \lambda I)\vec{v}_{(1,2)} = \vec{0} = \begin{pmatrix} 8 & 4 \\ -16 & -8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 8v_1 + 4v_2 = 0 \Rightarrow v_2 = -2v_1$. set $v_1 = \alpha$; $v_2 = -2\alpha$

$$\Rightarrow \vec{v} = \begin{pmatrix} \alpha \\ -2\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \vec{v}_{(1,2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Solve $(A - \lambda I)\vec{u} = \vec{v}$ for generalized eigenvector:

$$\begin{pmatrix} 8 & 4 \\ -16 & -8 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow 8u_1 + 4u_2 = 1$$

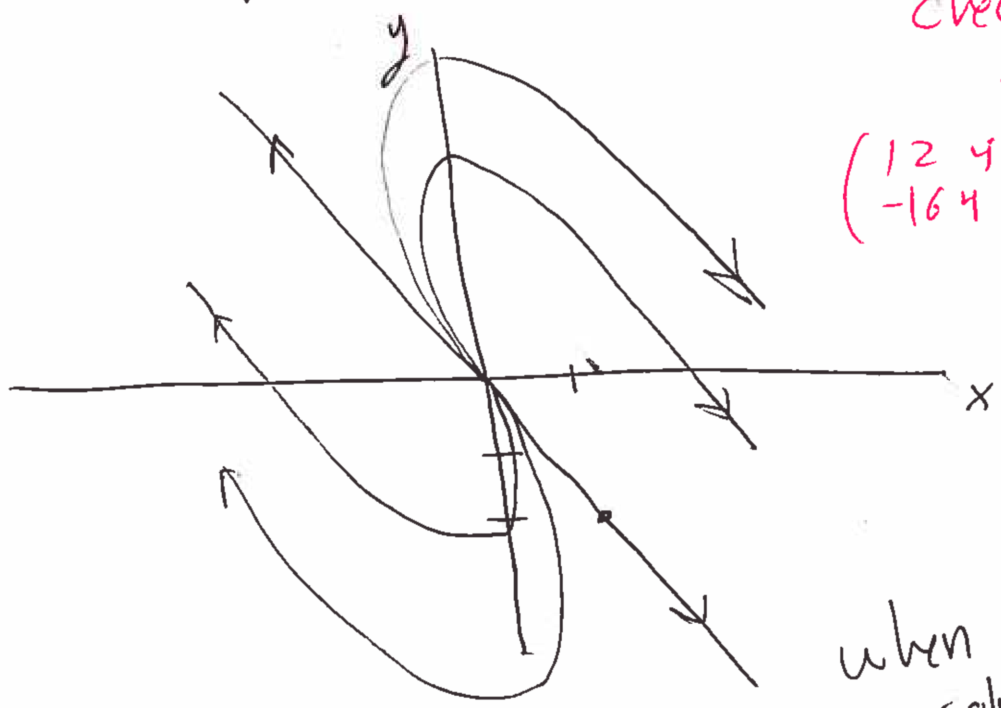
set $u_1 = 0 \Rightarrow u_2 = \frac{1}{4} \Rightarrow \vec{u} = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}$

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2\left[te^{\lambda t}\cancel{\vec{v}} + e^{\lambda t}\vec{u}\right]$$

$$= c_1 e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\Big[\underbrace{te^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{\text{this term dominates as } t \to \infty} + e^{4t}\begin{pmatrix} 0 \\ 1/4 \end{pmatrix}\Big]$$

Solution becomes parallel to $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ as $t \to \infty$.

Solution diverges from eq pt $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$ since eigenvalue is positive.



check tangent vec direction

$$\begin{pmatrix} 12 & 4 \\ -16 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ -16 \end{pmatrix}$$

$(1,0)$

Consistent with plot

when $c_2 = 0$
solution coincides with vector $\vec{v}$.

Integrating factor

$$\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow \mu(x) = e^{\int P(x)dx}$$

$$\Rightarrow \frac{d}{dx}[\mu(x)y(x)] = \mu(x)Q(x)$$

Ex) $y' - e^{x^2} = 2xy \Rightarrow y' - 2xy = e^{x^2}$

$p(x) = -2x$

Ex) $\vec{x}' = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -4 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

Take $\lambda_1 = 2i$:

$(A - \lambda_1 I)\vec{v}_{(1)} = \vec{0} \Rightarrow \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -2iv_1 - 4v_2 = 0 \Rightarrow v_1 = 2$; $v_2 = -i$

$\Rightarrow \vec{v}_{(1)} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -i \end{pmatrix}$

Take $\vec{X}_{comp}(t) = e^{\lambda_1 t}\vec{v}_{(1)} = e^{2it}\begin{pmatrix} 2 \\ -i \end{pmatrix} =$

$$= [\cos(2t) + i\sin(2t)]\begin{bmatrix} 2 \\ -i \end{bmatrix} =$$

$$= \begin{bmatrix} 2\cos(2t) + i\,2\sin(2t) \\ -i\cos(2t) - i^2\sin(2t) \end{bmatrix} \quad (\text{since } i^2 = -1)$$

$$= \begin{bmatrix} 2\cos(2t) + i\,2\sin(2t) \\ \sin(2t) - i\cos(2t) \end{bmatrix}$$

$\vec{X}_1(t) = \mathrm{Re}[\vec{X}_{comp}(t)]$ and $\vec{X}_2(t) = \mathrm{Im}[\vec{X}_{comp}(t)]$

$\Rightarrow \vec{X}_1(t) = \begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix}$; $\vec{X}_2(t) = \begin{bmatrix} 2\sin(2t) \\ -\cos(2t) \end{bmatrix}$

Thus,

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = c_1\begin{bmatrix} 2\cos(2t) \\ \sin(2t) \end{bmatrix} + c_2\begin{bmatrix} 2\sin(2t) \\ -\cos(2t) \end{bmatrix}$$

$$\vec{x}(0) = c_1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow c_1 = \frac{1}{2}; \quad c_2 = -1$$

$$\Rightarrow \vec{x}(t) = \begin{bmatrix} \cos(2t) \\ \frac{1}{2}\sin(2t) \end{bmatrix} - \begin{bmatrix} 2\sin(2t) \\ -\cos(2t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2t) - 2\sin(2t) \\ \frac{1}{2}\sin(2t) + \cos(2t) \end{bmatrix}$$

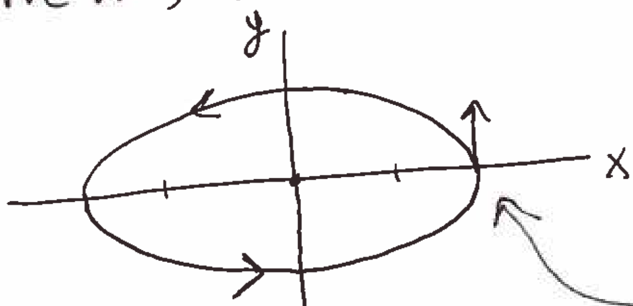
$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \Rightarrow \begin{array}{l} x(t) = \cos(2t) - 2\sin(2t) \\ y(t) = \frac{1}{2}\sin(2t) + \cos(2t) \end{array}$$

Notice that $|x(t)| \le |\cos(2t)| + 2|\sin(2t)| \le 1 + 2 = 3$

$|y(t)| \le \frac{1}{2}|\sin(2t)| + |\cos(2t)| \le \frac{1}{2} + 1 = \frac{3}{2}$

Hence, the trajectories form an ellipse wider in x direction.

stable center



points up

check tangent at $(3,0)$:

$\vec{x}'(t^*) = A\vec{x}(t^*)$ where

$\vec{x}(t^*) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$\Rightarrow \vec{x}'(t^*) = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

Ex)
$$\vec{x}\,' = \begin{pmatrix} -3 & 1 & -2 \\ 0 & -1 & -1 \\ 2 & 0 & 0 \end{pmatrix} \vec{x}$$
Find general solution.

$$|A-\lambda I| = \begin{vmatrix} -3-\lambda & 1 & -2 \\ 0 & -1-\lambda & -1 \\ 2 & 0 & -\lambda \end{vmatrix} =$$
equation for λ will be a cubic poly.

$$= (-3-\lambda)\begin{vmatrix} -1-\lambda & -1 \\ 0 & -\lambda \end{vmatrix} + (2)\begin{vmatrix} 1 & -2 \\ -1-\lambda & -1 \end{vmatrix}$$

$$= (-3-\lambda)\left[-\lambda(-1-\lambda)\right] + 2\left[-1 + 2(-1-\lambda)\right]$$

$$\left.\begin{aligned} &= -(3+\lambda)\lambda(1+\lambda) + 2\left[-1-2(1+\lambda)\right] \\ &= -(3+\lambda)\lambda(1+\lambda) - 2 - 4(1+\lambda) \\ &= (-3\lambda - \lambda^2)(1+\lambda) - 2 - 4 - 4\lambda \end{aligned}\right\}$$
ideally, we can factor this expression but in case we can't, we expand to a cubic equation.

$$= (-3\lambda - \lambda^2) + (-3\lambda^2 - \lambda^3) - 6 - 4\lambda$$

$$= -\lambda^3 - 4\lambda^2 - 7\lambda - 6 = 0 \Rightarrow \lambda^3 + 4\lambda^2 + 7\lambda + 6 = 0$$

By "inspection" see that $\lambda = -2$ is a root (or use software)

$$(-2)^3 + 4\cdot 4 - 14 + 6 = -8 + 16 - 14 + 6 = -22 + 22 = 0$$

Thus, we can factor out $(\lambda+2)$ to get:

$$\lambda^3 + 4\lambda^2 + 7\lambda + 6 = (\lambda+2)(\lambda^2 + 2\lambda + 3) = 0$$

so $\lambda_1 = -2$ and λ_2, λ_3 are roots of $(\lambda^2 + 2\lambda + 3) = 0$

$\lambda^2 + 2\lambda + 3 = 0 \Rightarrow \lambda = \frac{\cancel{-4}\ -2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2}$

$\lambda = -1 \pm \sqrt{2}\,i$. So the eigenvalues are:

$\lambda_1 = -2$; $\lambda_2 = -1 - \sqrt{2}\,i$; $\lambda_3 = -1 + \sqrt{2}\,i$

$$(A - \lambda_1 I)\vec{V}_{(1)} = \vec{0} \Rightarrow \begin{pmatrix} -1 & 1 & -2 \\ 0 & 1 & -1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Notice that since $\det(A - \lambda_1 I) = 0$, the rank of the 3×3 matrix is at most 2. It is exactly two since the first two rows are linearly independent.

So use for ex, 2nd and 3rd equations:

$$V_2 - V_3 = 0 \Rightarrow V_2 = V_3$$

$$2V_1 + 2V_3 = 0 \Rightarrow V_1 = -V_3 = -V_2$$

$$\Rightarrow \vec{V}_{(1)} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\vec{V}_{(2)}$ and $\vec{V}_{(3)}$ will be complex conjugates.

$(A - \lambda_2 I)\vec{V}_{(2)} = \vec{0}$ gives:

$$\begin{pmatrix} -2+\sqrt{2}\,i & 1 & -2 \\ 0 & \sqrt{2}\,i & -1 \\ 2 & 0 & 1+\sqrt{2}\,i \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Take 2nd and 3rd eqs:

$$\sqrt{2}\,i\,V_2 - V_3 = 0$$
$$2V_1 + (1+\sqrt{2}\,i)V_3 = 0$$

So $v_3 = \sqrt{2} i \, v_2$ and $v_1 = \frac{-(1+\sqrt{2}i)}{2} v_3$

$$= \frac{-(1+\sqrt{2}i)}{2} \sqrt{2} i \, v_2$$

Let $v_2 = \alpha$

$$= -\tfrac{1}{2}\left[\sqrt{2}i - \cancel{\sqrt{2}}\right] v_2$$

Then $v_3 = \sqrt{2} i \alpha$ and $v_1 = +\frac{\sqrt{2}}{2}\left[\sqrt{2} - i\right] \alpha$

$$\Rightarrow \tilde{\vec{v}}_{(2)} = \begin{pmatrix} \frac{\sqrt{2}}{2}(\sqrt{2}-i)\alpha \\ \alpha \\ \sqrt{2} i \alpha \end{pmatrix}$$

~~mult by~~ let $\alpha = 2$:

$$\vec{v}_{(2)} = \begin{pmatrix} 2-\sqrt{2}i \\ 2 \\ 2\sqrt{2}i \end{pmatrix} \Rightarrow \vec{v}_{(3)} = \begin{pmatrix} 2+\sqrt{2}i \\ 2 \\ -2\sqrt{2}i \end{pmatrix}$$

$$\vec{x}_{comp}(t) = e^{\lambda_2 t} \vec{v}_{(2)} = e^{(-1-\sqrt{2}i)t} \begin{pmatrix} 2-\sqrt{2}i \\ 2 \\ 2\sqrt{2}i \end{pmatrix} =$$

$$= e^{-t} e^{-\sqrt{2}it} \begin{pmatrix} 2-\sqrt{2}i \\ 2 \\ 2\sqrt{2}i \end{pmatrix}$$

$$= e^{-t}\left[\cos(\sqrt{2}t) - i\sin(\sqrt{2}t)\right] \begin{pmatrix} 2-\sqrt{2}i \\ 2 \\ 2\sqrt{2}i \end{pmatrix}$$

$$= e^{-t}\begin{bmatrix} 2\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \\ 2\cos(\sqrt{2}t) \\ 2\sqrt{2}\sin(\sqrt{2}t) \end{bmatrix} - i e^{-t} \begin{bmatrix} 2\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \\ 2\sin(\sqrt{2}t) \\ -2\sqrt{2}\cos(\sqrt{2}t) \end{bmatrix}$$

To get $\vec{x}_2, \vec{x}_3$ take real and imaginary parts:

$$\vec{X}_2(t) = Re\left[\vec{X}_{comp}(t)\right] = e^{-t}\begin{bmatrix} 2\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \\ 2\cos(\sqrt{2}t) \\ 2\sqrt{2}\sin(\sqrt{2}t) \end{bmatrix}$$

$$\vec{X}_3(t) = Im\left[\vec{X}_{comp}(t)\right] = -e^{-t}\begin{bmatrix} 2\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \\ 2\sin(\sqrt{2}t) \\ -2\sqrt{2}\cos(\sqrt{2}t) \end{bmatrix}$$

Since $\lambda_1 = -2$ is real, the first sol vector is:

$$\vec{X}_1(t) = e^{\lambda_1 t}\vec{V}_{(1)} = e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Hence, general solution is:

$$\vec{X}(t) = C_1\vec{X}_1(t) + C_2\vec{X}_2(t) + C_3\vec{X}_3(t)$$

Linearization of nonlinear systems

Suppose $\begin{cases} x'(t) = f(x,y) \\ y'(t) = g(x,y) \end{cases}$ is a nonlinear (*) system. Define the Jacobian matrix to be:

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

At an equilibrium point (x_E, y_E) of system (*), the stability characteristics of (*) at that point depend on the eigenvalues of

$$J(x_E, y_E) = \begin{bmatrix} f_x(x_E, y_E) & f_y(x_E, y_E) \\ g_x(x_E, y_E) & g_y(x_E, y_E) \end{bmatrix}$$

This is a great result since it allows us to characterize equilibrium points of <u>nonlinear</u> systems using Jacobian matrix which is usually not difficult to compute.

See p. 437, Table 7.2.1 in book.

Ex We analyze the equilibrium points of the system corresponding to the nonlinear equation

$$x'' + x - x^2 - 2x^3 = 0 \quad (1)$$

Let $y = x'$. Then $y' = x'' = -x + x^2 + 2x^3$

System corresponding to eq (1) is:

$$\begin{aligned} x' &= y \\ y' &= -x + x^2 + 2x^3 \end{aligned} \Rightarrow J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

where $f(x,y) = y$ and $g(x,y) = -x + x^2 + 2x^3$

$$\Rightarrow J(x,y) = \begin{bmatrix} 0 & 1 \\ -1+2x+6x^2 & 0 \end{bmatrix}$$

Next, we find the equilibrium points:

$$y = 0 = -x + x^2 + 2x^3$$

$$\Rightarrow x(2x^2 + x - 1) = 0 \Rightarrow x = 0, \; 2x^2 + x - 1 = 0$$

$$2x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$$

$$\left.\right\} \begin{aligned} x &= 0 \\ x &= -1 \\ x &= \tfrac{1}{2} \end{aligned}$$

So the equilibrium points are

$(0,0)$, $(-1,0)$, $(\frac{1}{2},0)$

We analyze the stability properties of the system at each equilibrium point by finding the eigenvalues of the Jacobian matrix at each of the points.

(a) At $(0,0)$, $J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ no real part purely imaginar

$$|J-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & \cancel{0}-\lambda \end{vmatrix} = \lambda^2+1=0 \Rightarrow \lambda = \pm i$$

Borderline case.

This corresponds to a stable center for the linearized system $\Rightarrow$ center, stable/unstable spiral for nonlinear system.

(b) At $(\frac{1}{2},0)$, $J(\frac{1}{2},0) = \begin{bmatrix} 0 & 1 \\ \frac{3}{2} & 0 \end{bmatrix}$

$$|J-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ \frac{3}{2} & -\lambda \end{vmatrix} = \lambda^2 - \frac{3}{2} = 0 \Rightarrow \lambda = \pm\sqrt{\frac{3}{2}}$$

The eigenvalues are real of opposite signs this corresponds to (unstable) saddle point.

(c) At $(-1,0)$, $J(-1,0) = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

$$|J-\lambda I| = \lambda^2-3=0 \Rightarrow \lambda = \pm\sqrt{3} \Rightarrow \text{saddle point}$$