Equations of Glider Motion in 2D

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1 Introduction

This document presents the equations of motion for a glider in 2D under the influence of gravity and a simple air resistance model. The equations are derived from Newton's second law of motion, considering the angle of flight, and are then simplified to first-order form suitable for numerical solving using an ODE solver.

2 Forces Acting on the Glider

The motion of the glider is influenced by gravity, lift, and drag forces. Below are the horizontal and vertical components of these forces.

2.1 Gravity

Gravity acts vertically downward with force:

- **Vertical component:** $F_{g_y} = -mg$
- **Horizontal component:** $F_{g_x} = 0$

2.2 Lift Force (L)

The lift force acts perpendicular to the velocity vector ($\vec{v}$) and is calculated as:

$$L = \frac{1}{2}\rho v^2 S C_L$$

Decomposing lift into components:

- **Vertical component:** $L_y = L\cos\theta$
- **Horizontal component:** $L_x = L\sin\theta$

2.3 Drag Force (D)

The drag force acts opposite to the velocity vector and is calculated as:

$$D = \frac{1}{2}\rho v^2 S C_D$$

Decomposing drag into components:

- **Vertical component:** $D_y = -D \sin\theta$
- **Horizontal component:** $D_x = -D \cos\theta$

2.4 Diagram of Forces

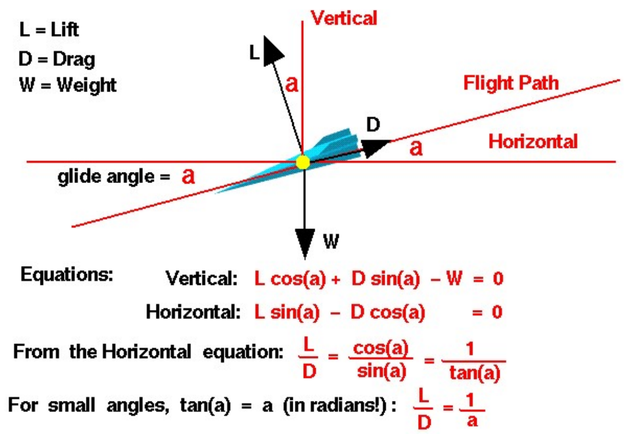

Figure 1: Diagram showing the forces acting on the glider and their components.

3 Flight Path Angle and Angle of Attack

3.1 Flight Path Angle (γ)

The flight path angle (γ) is the angle between the horizontal axis and the velocity vector of the glider. It indicates the direction of the flight path relative to the ground.

3.2 Angle of Attack (α)

The angle of attack (α) is the angle between the chord line of the glider's wing and the velocity vector. It determines the aerodynamic forces, such as lift and drag.

3.3 Relation Between Angles

For simplicity, we often assume that the flight path angle is equal to the angle of attack ($\gamma = \alpha$). This assumption allows us to use a single angle (θ) in the equations of motion.

4 Equations of Motion

According to Newton's second law, the sum of the forces acting on an object is equal to the mass of the object multiplied by its acceleration:

$$\vec{F} = m\vec{a} \tag{1}$$

For the glider, we can write the equations of motion in the horizontal and vertical directions as follows:

4.1 Horizontal Motion

The forces acting in the horizontal direction are the horizontal components of the lift and drag forces:

$$\begin{aligned} F_x &= L \sin\theta - D \cos\theta && (2)\\ &= \frac{1}{2}\rho v^2 S C_L \sin(\theta) - \frac{1}{2}\rho v^2 S C_D \cos(\theta) && (3)\\ &= m\frac{d^2x}{dt^2} && (4) \end{aligned}$$

Rearranging, we get:

$$m\frac{d^2x}{dt^2} = \frac{1}{2}\rho v^2 S C_L \sin(\theta) - \frac{1}{2}\rho v^2 S C_D \cos(\theta) \tag{5}$$

4.2 Vertical Motion

The forces acting in the vertical direction are the gravitational force and the vertical components of the lift and drag forces:

$$\begin{aligned} F_y &= L \cos\theta - mg - D \sin\theta && (6)\\ &= \frac{1}{2}\rho v^2 S C_L \cos\theta - \frac{1}{2}\rho v^2 S C_D \sin\theta - mg && (7)\\ &= m\frac{d^2y}{dt^2} && (8) \end{aligned}$$

Rearranging, we get:

$$m\frac{d^2y}{dt^2} = \frac{1}{2}\rho v^2 S C_L \cos\theta - \frac{1}{2}\rho v^2 S C_D \sin\theta - mg \tag{9}$$

5 Derivation of Angular Motion Equation

To understand the angular motion of the glider, we need to consider the forces that act perpendicular to the velocity vector and cause a change in the direction of motion.

The lift force L and the drag force D can be decomposed into components parallel and perpendicular to the velocity vector. The components perpendicular to the velocity vector are responsible for changing the direction of the glider's motion.

- Perpendicular component of lift: $L\cos(\theta)$
- Perpendicular component of drag: $D\sin(\theta)$

The net perpendicular force is:

$$F_{\perp} = L\cos(\theta) + D\sin(\theta)$$

This perpendicular force causes an angular acceleration $\frac{d\theta}{dt}$, which is related to the tangential acceleration by:

$$F_{\perp} = ma_{\perp}$$

where $a_{\perp}$ is the centripetal acceleration. For an object moving with a linear velocity v along a curved path, the centripetal acceleration $a_{\perp}$ is given by:

$$a_{\perp} = \omega v$$

where $\omega = \frac{d\theta}{dt}$ is the angular velocity. Thus, we get:

$$a_{\perp} = v\frac{d\theta}{dt}$$

Substituting this into the force equation, we get:

$$mv\frac{d\theta}{dt} = L\cos(\theta) + D\sin(\theta)$$

Rearranging, we get:

$$\frac{d\theta}{dt} = \frac{1}{mv}\left(L\cos(\theta) + D\sin(\theta)\right)$$

6 Simplification to First-Order Form

To solve these second-order differential equations numerically, we convert them to first-order form by introducing the following variables:

$$u = \frac{dx}{dt},$$
$$w = \frac{dy}{dt}.$$

The speed of the glider is:

$$v = \sqrt{u^2 + w^2} \tag{10}$$

The angle of flight is derived from the velocity components:

$$\theta = \arctan\left(\frac{w}{u}\right) \tag{11}$$

Substituting u and w into the equations of motion, we get:
1. – Horizontal position – :

$$\frac{dx}{dt} = u$$

2. – Horizontal velocity – :

$$\frac{du}{dt} = \frac{1}{m}\left(\frac{1}{2}\rho v^2 SC_L \sin(\theta) - \frac{1}{2}\rho v^2 SC_D \cos(\theta)\right)$$

3. – Vertical position – :

$$\frac{dy}{dt} = w$$

4. – Vertical velocity – :

$$\frac{dw}{dt} = \frac{1}{m}\left(\frac{1}{2}\rho v^2 SC_L \cos(\theta) - \frac{1}{2}\rho v^2 SC_D \sin(\theta) - mg\right)$$

5. – Angle of flight – :

$$\frac{d\theta}{dt} = \frac{1}{mv}\left(\frac{1}{2}\rho v^2 SC_L \cos(\theta) + \frac{1}{2}\rho v^2 SC_D \sin(\theta)\right)$$

These first-order differential equations can be integrated from a proper set of initial conditions using standard numerical techniques such as e.g. Runge Kutta methods.

7 Interpretation and Initial Conditions

This system of equations describes the dynamics of a glider moving in two dimensions under the influence of gravity, lift, and drag. The equations account for the forces acting on the glider and their components, and how these forces influence the glider's velocity, position, and flight path angle over time.

The variables u and w represent the horizontal and vertical components of the velocity, respectively:

$$u = \frac{dx}{dt}, \quad w = \frac{dy}{dt}$$

These components are used to calculate the total speed of the glider:

$$v = \sqrt{u^2 + w^2}$$

The angle of flight (θ) is derived from the velocity components:

$$\theta = \arctan\left(\frac{w}{u}\right)$$

Using the definitions of lift and drag,

$$L(v) = \frac{1}{2}\rho v^2 S C_L$$

$$D(v) = \frac{1}{2}\rho v^2 S C_D$$

, we can rewrite the equations of motion as:

1. – Horizontal position – :

$$\frac{dx}{dt} = u$$

2. – Horizontal velocity – :

$$\frac{du}{dt} = \frac{1}{m}\left(L(v)\sin(\theta) - D(v)\cos(\theta)\right)$$

3. – Vertical position – :

$$\frac{dy}{dt} = w$$

4. – Vertical velocity – :

$$\frac{dw}{dt} = \frac{1}{m}\left(L(v)\cos(\theta) - D(v)\sin(\theta) - mg(y)\right)$$

5. – Angle of flight – :

$$\frac{d\theta}{dt} = \frac{1}{mv}\left(L(v)\cos(\theta) + D(v)\sin(\theta)\right)$$

7.1 Initial Conditions

Appropriate initial conditions should reflect realistic values for the glider's initial state:
– Initial horizontal velocity (u) – : Should be a positive value indicating the initial forward speed of the glider. For example, $u = 15$ m/s.
– Initial vertical velocity (w) – : Can be positive or negative depending on the initial climb or descent. For example, $w = 0$ m/s (level flight) or $w = -5$ m/s (initial descent).
– Initial angle of flight (θ) – : Should be a small positive or negative value, indicating a slight upward or downward angle. For example, $\theta = 0.1$ radians (slight climb) or $\theta = -0.1$ radians (slight descent).
– Initial horizontal position (x) – : Typically set to 0, representing the starting point. For example, $x = 0$ m.
– Initial vertical position (y) – : Should be a positive value representing the initial altitude of the glider. For example, $y = 1000$ m.

7.2 Effect of Altitude on Gravity

At high altitudes, the acceleration due to gravity (g) decreases with increasing altitude (y). This can be modeled using the following equation:

$$g(y) = g_0 \left(\frac{R}{R + y} \right)^2$$

where: - g_0 is the standard acceleration due to gravity at sea level ($9.81\,\mathrm{m/s^2}$), - R is the radius of the Earth ($6.371 \times 10^6\,\mathrm{m}$), - y is the altitude above sea level.

Incorporating this into the vertical velocity equation, we get:

$$\frac{dw}{dt} = \frac{1}{m} \left(L(v) \cos(\theta) - D(v) \sin(\theta) - mg(y) \right)$$

8 Conclusion

We have derived the equations of motion for a glider in 2D under the influence of gravity and air resistance, considering the angle of flight and incorporating the angular velocity relationship. These equations have been simplified to first-order form, making them suitable for numerical solving using an ODE solver. This model provides a basic understanding of the glider's dynamics which can be implemented with a basic ODE solver and can be extended for more complex simulations.