Final Exam Solutions

I)

$$\text{sort(males)} = \{\overset{1}{61}, \overset{2}{67}, \overset{3}{68}, \overset{4}{71}, \overset{5}{71}, \overset{6}{72}, \overset{7}{72}, \overset{8}{72}, \underset{9}{74}, \underset{10}{79}\}$$

$$L_{25} = \left(\frac{25}{100}\right) \times 10 = 2.5$$

$$Q_1 = sm[3] = 68$$

$$L_{50} = \left(\frac{50}{100}\right) \times 10 = 5$$

$$Q_2 = \frac{sm[5] + sm[6]}{2} = 71.5$$

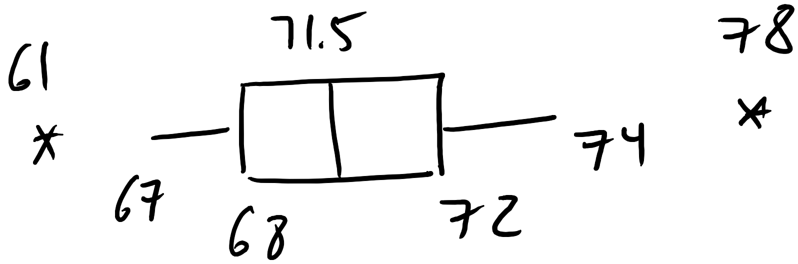
$$L_{75} = \left(\frac{75}{100}\right) \times 10 = 7.5$$

$$Q_3 = sm[8] = 72$$

$$IQR = Q_3 - Q_1 = 4$$

$$Q_1 - 1.5IQR = 62 \; ; \; Q_3 + 1.5IQR = 78$$

Thus, 61 and 79 are outliers.



sort(females)

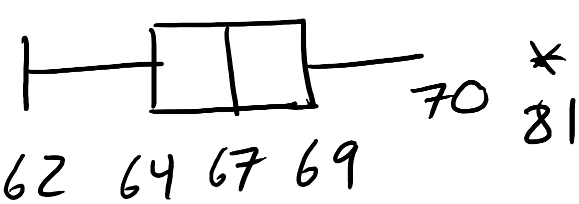
$$= \{\overset{1}{62}, \overset{2}{62}, \overset{3}{64}, \overset{4}{64}, \overset{5}{67}, \overset{6}{67}, \overset{7}{69}, \overset{8}{69}, \overset{9}{70}, \overset{10}{81}\}$$

$Q_1 = sf[3] = 64$

$$Q_2 = \frac{sf[5] + sf[6]}{2} = 67$$

$Q_3 = sf[8] = 69$

$IQR = Q_3 - Q_1 = 5$



$Q_1 - 1.5\,IQR = 56.5$

$Q_3 + 1.5\,IQR = 76.5$

81 is an outlier

$H_0: \mu_1 = \mu_2 \qquad (\mu_1 - \mu_2 = 0)$

$H_1: \mu_1 > \mu_2 \qquad (\mu_1 - \mu_2 > 0)$

$$t_s = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$\bar{x}_1 = 70.7 \;;\; \bar{x}_2 = 67.5 \;;\; n_1 = n_2 = 10$

$s_1 = 4.72 \;;\; s_2 = 5.56$

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$\approx \frac{478.6}{18} \approx 26.59$$

$$t_s = \frac{70.7 - 67.5}{\sqrt{26.59\left(\frac{2}{10}\right)}} \approx 1.39$$

$t_{\alpha, n_1+n_2-2} = t_{0.05, 18} = 1.734$

(using table or qt(1-0.05, 18) in R)

since $t_s < t_{\alpha, n_1+n_2-2}$

we can't reject H_0 at $\alpha = 0.05$ level.

II) We can subtract 1000 from each element of the set without changing the standard deviation.

$$S - 1000 = \{-.99, .78, 1.02, -1.22, 0.98, -0.75, 0.56, -1.25, 1.78, -0.24\} = X$$

$$sd(x) = \sqrt{\frac{1}{n-1} \sum (x_i - \bar{x})^2} \approx 1.09 = s$$

CI is of form:

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}$$

$\alpha = 0.1$; $n-1=9$

$\chi^2_{0.05,9} \approx 16.92$ (table or qchisq(1-0.05, 9))

$\chi^2_{0.95,9} \approx 3.33$

90% CI:

$$\frac{9\cdot(1.18)}{16.92} < \sigma^2 < \frac{9\cdot(1.18)}{3.33}$$

$\Rightarrow$ $0.63 < \sigma^2 < 3.19$

$\Rightarrow 0.79 < \sigma < 1.79$

Conduct hypothesis test for population variance:

$H_0: \sigma = 2 = \sigma_0$

$H_1: \sigma > 2$

$$\text{statistic} = \frac{(n-1)s^2}{\sigma_0^2}$$

$$\text{decision rule}: \chi_{test} < \chi^2_{1-\alpha, n-1}$$
$$= \chi^2_{0.9, 9}$$

(b) $$P(4\text{ aces}) = \frac{{}_4C_4 \times {}_{48}C_1}{{}_{52}C_5}$$

4 of 4 aces, 1 other card of 48 remaining

(c) $P(3 \text{ one suit, } 2 \text{ of another})$

$$= \frac{4({}_{13}C_3) \times 3({}_{13}C_2)}{{}_{52}C_5}$$

4 suits for first 3 cards, then next 2 cards must be from 3 remaining suits.

III) Roll a die

sample space: $\{1, 2, \dots, 6\}$

A: get even number $\{2, 4, 6\}$.

B: get a number less than or equal to 4.

$\{1, 2, 3, 4\}$

$P(A) = \frac{1}{2}$; $P(B) = \frac{4}{6} = \frac{2}{3}$

(a) since $A \cap B = \{2, 4\} \neq \phi$

the events are <u>not</u> mutually exclusive

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Notice that,

$P(A \cap B) = P(A)P(B)$ by independence in this case.

(b) Yes the events are independent

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/3}{2/3} = \frac{1}{2} = P(A)$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3} = P(B)$$

(c) $B^c = \{5,6\}$ (c denotes set complement)

$A \cap B^c = \{6\} \Rightarrow P(A\cap B^c) = \frac{1}{6}$

$A^c = \{1,3,5\} \Rightarrow A^c \cap B^c = \{5\}$

$P(A^c \cap B^c) = \frac{1}{6}$

(IV) Define two events

T : patient tests positive

D : patient has the disease

Apply Baye's rule

$$P(D|T) = \frac{P(T|D)\,P(D)}{P(T|D)\,P(D) + P(T|D')\,P(D')}$$

where $D' = D^c$: patient does not have disease

Now $P(T|D) = 1$ (rate of false negatives zero)

and $P(D) = \frac{2}{2000} = 0.001$. $P(T|D') = 0.05$ (false positives).

$$\Rightarrow P(D|T) = \frac{(1)(0.001)}{(1)(0.001) + (0.05)(.999)} \approx 0.02$$

That's not very accurate!

Now if $P(T|D') = 0.001$, then

$$P(D|T) = \frac{(1)(0.001)}{(1)(0.001) + (0.001)(.999)} \approx 0.50$$

This is a better result.

The probabilities are low because the rate of disease in the population is low.

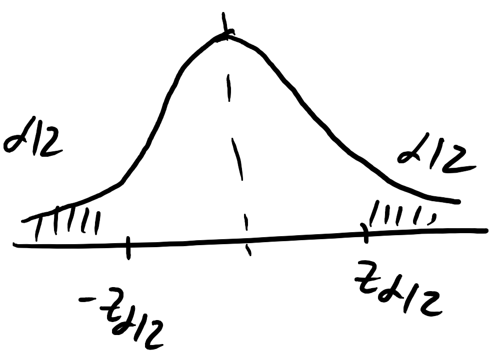
(V) (a)

$H_0: p = \frac{1}{2}$ (fair coin)

$H_1: p \neq \frac{1}{2}$ (unfair coin)

Notice that the test is two sided.

$\alpha = 0.05 \Rightarrow \alpha/2 = 0.025$



$z_{0.025} = 1.96$

Let x = # of heads

we reject H_0 for $|z_s| > z_{\alpha/2} = 1.96$

$$z_s = \frac{\overset{= n\hat{p}}{x} - \overset{= np_0}{450}}{\sqrt{npq}} = \frac{x-450}{\sqrt{800\frac{1}{2}\frac{1}{2}}} = \frac{x-450}{14.14}$$

$$z_s > 1.96 \Rightarrow \frac{x-450}{14.14} > 1.96$$

$$\Rightarrow x > 477.72 \Rightarrow \underline{x \geq 478}$$

$$z_s < -1.96 \Rightarrow \frac{x-450}{14.14} < -1.96$$

$\Rightarrow X - 450 < -27.72$

$\Rightarrow X < 422.88 \Rightarrow \underline{X \leq 422}$

So either less than or equal to 422 heads of 800 or greater than or equal to 478 heads will yield a rejection.

(b)

If 460 heads are obtained in 800 tosses and $H_1: p > \frac{1}{2}$ (Here told test one sided)

p-value $= P(B \geq 460)$ where $X \sim \text{Binomial}(800, \frac{1}{2})$
(evidence as or more contradictory to Null hypothesis)

$P(B \geq 460) \approx P(X \geq 455.5)$ where $X \sim \text{Normal}(np, \sqrt{npq})$

$$= P\left(\frac{X - np}{\sqrt{npq}} \geq \frac{455.5 - 400}{\sqrt{800 \frac{1}{2} \frac{1}{2}}} \right) = P(Z > 3.92)$$

$$= 0.5 - P(0 < Z < 3.92) \approx 4.43 \times 10^{-5}$$

Since p-value $< \alpha = 0.01$ we reject H_0.

(VI)

(a) 4% of chargers defective $\Rightarrow p = 0.04$

$B \sim \text{Binomial}(30, 0.04)$

$$P(B \leq 1) = P(B=0) + P(B=1)$$

$$= \binom{30}{0}(.04)^0(0.96)^{30-0} + \binom{30}{1}(.04)^1(0.96)^{29}$$

$$\approx 0.661$$

Using Poisson approximation, $\lambda = np = 30(.04) = 1.2$

$$P(B \leq 1) \approx P(X_p \leq 1) = P(X_p = 0) + P(X_p = 1)$$

$$= e^{-1.2}\left[\frac{(1.2)^0}{0!} + \frac{(1.2)^1}{1}\right] \approx 0.663$$

(b) We can consider an infinite # of cards with same proportions as in a single deck.

$$P(\text{jack}) = \frac{4}{52} = \frac{1}{13}$$

Let X : r.v. counting # cards to draw in order to get jack

Then $X \sim \text{Geometric}\left(\frac{1}{13}\right)$

$$P(X = n) = \left(\frac{12}{13}\right)^{n-1}\left(\frac{1}{13}\right)$$

$$P(X>12) = P(X \geq 13) = \sum_{n=13}^{\infty} \left(\frac{12}{13}\right)^{n-1}\left(\frac{1}{13}\right)$$

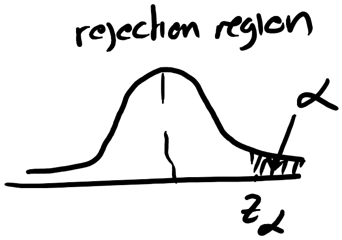
$$= \frac{1}{13} \sum_{n=3}^{\infty} \left(\frac{12}{13}\right)^{n-1} = \frac{1}{13} \frac{(12/13)^{12}}{1-12/13} = \left(\frac{12}{13}\right)^{12} \approx 0.38$$

(VII) (a) In one sample of 50 pipes, we have $s=200$.

Since $n>30$, we can use s as an estimate of σ in our estimate of type II error.

$H_0 : \mu = 2400$ vs $H_1 : \mu > 2400$

Let $\alpha = 0.05$ (any typical value is fine for estimate)



$z_{0.05} = 1.645$

Let us find $\bar{x}_0$, the largest value of $\bar{x}$ that supports H_0 (any larger value will put us in rejection region).

$$\bar{x}_0 = \mu_0 + z_\alpha \sigma_{\bar{x}} = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \approx \mu_0 + z_\alpha \frac{s}{\sqrt{n}}$$

$$= 2400 + 1.645\left(\frac{200}{\sqrt{50}}\right) \approx 2446.5$$ (we accept H_0 when $\bar{x} < \bar{x}_0$)

P(type II error) = P(accept H_0 when it is false)

= P(accept H_0 when H_1 true)

We assume as stated that $\mu_1 = 2450$, so:

$$P(\text{type II error}) = P(\text{accept } H_0 \mid \mu = 2450)$$

$$= P(\bar{x} < 2446.5 \mid \mu = 2450)$$

$$= P\left(\frac{\bar{x} - \mu}{\sigma_{\bar{x}}} < \frac{2446.5 - \mu}{\sigma_{\bar{x}}} \right) \qquad \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \approx \frac{s}{\sqrt{n}}$$

$$= P\left(z < \frac{2446.5 - 2450}{(200/\sqrt{50})} \right)$$

$$\approx P(z < .76) = 0.5 + P(0 < z < .76)$$

$$\approx 0.5 + .276 = .776 = \beta$$

power of test $= 1 - \beta = 0.224$, if this μ_1 is true.

(b)

$H_0: M = 5250$

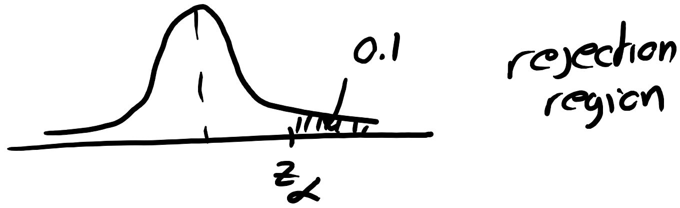
$H_1: M > 5250$

$S = 13$ and $n = 20$

Large scale median test statistic:

$$z_s = \frac{(S - 0.5) - 0.5n}{\underbrace{\sqrt{(0.5)(0.5)n}}_{\sqrt{npq}}} = \frac{(13 - 0.5) - 10}{0.5\sqrt{20}} \approx 1.18$$

$\alpha = 0.1$



$z_{0.1} = 1.28 \qquad \left(P(0 < z < z_\alpha) = 0.5 - 0.1 = 0.4 \right)$

since $z_s < z_\alpha$ we do not reject H_0 at this confidence level.