Exam 2 solutions

I) 3 coin tosses, 2 outcomes per toss

X : r.v. counting # tails in 3 tosses

$2^3 = 8$ possible outcomes :

$\{\underset{1}{HHH}, \underset{2}{HTH}, \underset{3}{HHT}, \underset{4}{HTT}, \underset{5}{TTH}, \underset{6}{THT}, \underset{7}{THH}, \underset{8}{TTT}\}$

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\sum P(X=x) = 1$

$$P(X \geq 2) = P(X=2) + P(X=3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

$$E[X] = \sum x P(X=x) = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$

$$Var[X] = \sum (x - E[X])^2 P(X=x)$$

$$= (0-1.5)^2\left(\frac{1}{8}\right) + (1-1.5)^2\left(\frac{3}{8}\right) + (2-1.5)^2\left(\frac{3}{8}\right)$$

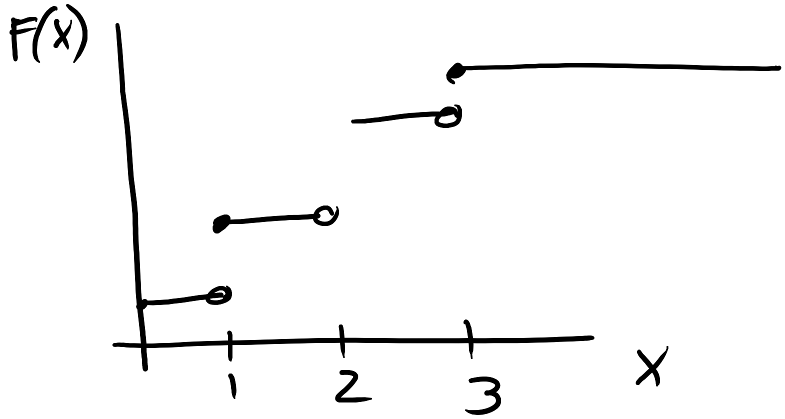
$$+ (3-1.5)^2\left(\frac{1}{8}\right) = 0.75$$

notice that X is actually <u>Binomial</u>.

Hence, $E[X] = np = 3\frac{1}{2} = 1.5$

$Var[X] = npq = 3\frac{1}{2}\frac{1}{2} = 0.75$

CDF : $F(x) = P(X \leq x)$ looks like a step function



$F(x) = 1$
for $x \geq 3$.

II) A) $3!\,6!\,N = 9! \Rightarrow N = \frac{9!}{3!\,6!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2}$
$= 84$

B) 7 chemists 5 poets

committee : 3 chemists, 2 poets

no restriction: $\binom{7}{3} \times \binom{5}{2} = \frac{7!}{3!\,4!} \times \frac{5!}{2!\,3!} = 350$ possibilities

restriction: two particular chemists cannot be on (must choose from remaining 5), one particular poet must be on (just one poet left to choose out of 4 remaining).

$$\binom{5}{3} \times \binom{4}{1} = \frac{5!}{3!\,2!} \times \frac{4!}{1!\,3!} = 40$$

(c) Choose 5 cards out of 52 card deck

Order here does not matter.

$$P(4 \text{ aces}, 1 \text{ jack}) = \frac{{}_4C_4 \times {}_4C_1}{{}_{52}C_5}$$

$$= \frac{1}{649{,}740}$$ very small

$$P(\text{no ace}) = \frac{{}_{48}C_5}{{}_{52}C_5}$$

$$P(\text{at least one ace}) = 1 - \frac{{}_{48}C_5}{{}_{52}C_5} = \frac{18{,}472}{54{,}145}$$

III) Use Baye's rule. Define events:

I: choose tea from box I

II: choose tea from box II

G: you drank green tea

B: you drank black tea

two dice $\Rightarrow$ sum 3 : $\{(1,2),(2,1)\} \Rightarrow P(\text{sum } 3) = \frac{2}{36}$

sum 6 : $\{(3,3),(1,5),(5,1),(2,4),(4,2)\}$

$P(\text{sum } 6) = \frac{5}{36}$

sum 12 : $\{(6,6)\} \Rightarrow P(\text{sum } 12) = \frac{1}{36}$

$$P(\text{sum } 3 \text{ or sum } 6 \text{ or sum } 12) = \frac{2}{36} + \frac{5}{36} + \frac{1}{36} = \frac{8}{36} = P(I)$$

(A)

$$P(G) = P(I)\,P(G|I) + P(II)\,P(G|II)$$

$$= \frac{8}{36} \times \frac{2}{8} + \frac{28}{36} \times \frac{15}{40} \approx 0.347$$

6 black 2 green	25 black 15 green
Box I 8 total	Box II 40 total

(B) P(your tea came from Box I given you drank black)

$$= P(I|B) = \frac{P(I)\,P(B|I)}{P(I)\,P(B|I) + P(II)\,P(B|II)}$$

$P(I) = \frac{8}{36}$; $P(II) = \frac{28}{36}$

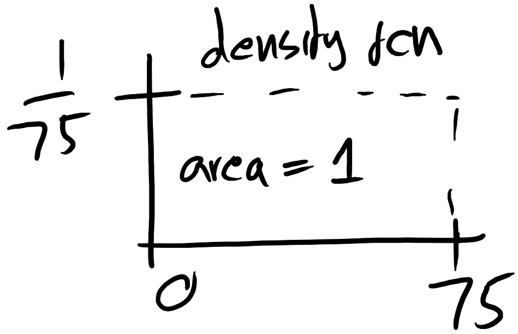
$P(B|I) = \frac{6}{8}$; $P(B|II) = \frac{25}{40}$

$$\Rightarrow P(I|B) = \frac{(8/36) \times (6/8)}{(8/36) \times (6/8) + (28/36) \times (25/40)}$$

$$\approx \frac{0.16}{0.65} \approx 0.246$$

(IV) $X \sim \text{Uniform}(0, 75)$

(A) $E(x) = \frac{75+0}{2} = 37.5$ min



$$F(x) = P(X \le x)$$
$$= \frac{1}{75} x \quad \text{CDF}$$

$$P(X < 15) = P(X \le 15) = \frac{1}{75} 15 = \frac{1}{5}$$

$$P(X > 30) = 1 - P(X < 30) = 1 - F(30)$$
$$= 1 - \frac{1}{75} 30 = \frac{45}{75} = \frac{3}{5}$$

(B) X: r.v. recording # of lottery tickets bought before 100 dollar win occurs.

Then $X_g \sim Geo(p)$ with $p = 0.001$.

each ticket @ \$2. Need to buy more than 300 tickets to spend > \$600.

$$P(X_g > 300) = 1 - P(X_g \leq 300)$$

Recall Geo CDF:

$$F(x) = P(X_g \leq x) = 1 - q^{\lfloor x \rfloor} = 1 - (1-p)^{\lfloor x \rfloor}$$

$$P(X_g > 300) = 1 - [1 - 0.999^{300}]$$

$$\approx 1 - 0.259 = .741$$

$$E[X_g] = \frac{1}{p} = \frac{1}{0.001} = 1000 \text{ tickets}$$

is the expected value needed to buy to win \$100. This costs \$2000.

(c) Suppose we buy $n=50$ tickets.

Let $p = 0.001$ (win probability).

X = # winning tickets amongst n

Then $X \sim \text{Binomial}(p)$.

$$P(X<2) = P(X=0) + P(X=1)$$

$$= \binom{50}{0}(0.001)^0(0.999)^{50} + \binom{50}{1}(0.999)^{49}(0.001)$$

$$\approx .951 + 0.0476 \approx 0.9986$$

The probability that we have less than two winning tickets is very high.

(d) Now we use Poisson distribution.

$P(X_b \geq 2)$ (at least two winning tickets)

$$\approx P(X_p \geq 2) = 1 - P(X_p < 2) = 1 - P(X_p \leq 1)$$

$$= 1 - [P(X_p = 0) + P(X_p = 1)]$$

$$= 1 - \left[\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!}\right]$$

$= 1 - [e^{-\lambda} + e^{-\lambda}\lambda] = 1 - e^{-\lambda}(1+\lambda)$

$\lambda = np = 50(0.001) = 0.05$

$\Rightarrow P(X_p \geq 2) = 1 - e^{-0.05}(1+0.05) \approx 0.0012$

The probability of having multiple winning tickets amongst the 50 is small.

(V) Since $n \geq 30$ the CLT applies.

(A) $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$. Notice that for each sample the value of the sample mean $\bar{X}_i$ may differ, but all values taken together form approx. a normal distribution.

$E[\bar{X}] = \mu = 6.8$

$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3.0}{\sqrt{30}} \approx 0.548$

(B) $P(66.6 < \bar{x} < 68.5) =$

$$= P\left(\frac{66.6 - \mu}{\sigma/\sqrt{n}} < \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < \frac{68.5 - \mu}{\sigma/\sqrt{n}} \right)$$

$$= P\left(\frac{66.6 - 68}{0.548} < z < \frac{68.5 - 68}{0.548} \right)$$

$$\approx P(-2.555 < z < 0.912)$$

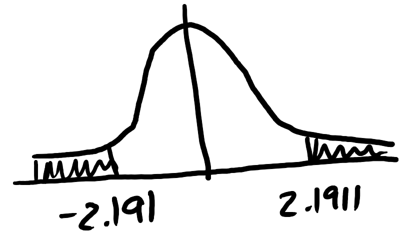
$$= P(0 < z < 2.555) + P(0 < z < 0.912)$$

$$\approx .4946 + .3186 = .8132$$

samples approx for which $\bar{x}$ in above range $\approx \lfloor .8132 \times 80 \rfloor = 65$ samples of 80.

$$P(\bar{x} < 66.8) = P\left(z < \frac{66.8 - 68}{3/\sqrt{30}}\right)$$

$$\approx P(z < -2.191)$$



$$= 0.5 - P(0 < z < 2.191)$$

$$\approx 0.5 - 0.4857 \approx 0.0143$$

samples $\approx \lfloor 0.0143 \times 80 \rfloor = 1$ only one sample of 80 expected to have such mean.