Exam 3 solutions

1) $n = 40$; $p = 0.5$

$S_n = \#$ heads in 40 flips (Binomial r.v.)

(a) $P(17 \le S_n \le 21) =$

$= P(S_n \le 21) - P(S_n \le 16)$ [CDFs]

$=$ pbinom(21, 40, .5) − pbinom(16, 40, .5)

≈ 0.548

(b) without continuity correction:

$$P(17 \le S_n \le 21) = P\left(\frac{17-np}{\sqrt{npq}} \le \frac{S_n - np}{\sqrt{npq}} \le \frac{21-np}{\sqrt{npq}}\right)$$

$$= P\left(\frac{17-20}{3.16} \le Z \le \frac{21-20}{3.16}\right) \quad (p = q = 0.5)$$

$$= P(-0.95 \le Z \le 0.32)$$

$$= P(Z \le 0.32) - P(Z < -0.95) \quad \text{(CDF)}$$

$$= P(Z \le 0.32) - P(Z \le -0.95)$$ since Z is continuous

$$= \text{pnorm}(0.32) - \text{pnorm}(-0.95) \approx 0.45$$

(c) with continuity correction

$$P(17 \le S_n \le 21) \approx P(16.5 \le X \le 21.5)$$

where X is normal r.v.

$$= P\left(\frac{16.5 - np}{\sqrt{npq}} \le Z \le \frac{21.5 - np}{\sqrt{npq}} \right)$$

$$= P\left(\frac{16.5 - 20}{3.16} \le Z \le \frac{21.5 - 20}{3.16} \right)$$

$$\approx P(-1.10 \le Z \le 0.475)$$

$$= \text{pnorm}(0.475) - \text{pnorm}(-1.10) \approx 0.547$$

Notice that the continuity correction has a significant effect on the result, bringing it much closer to the true Binomial probability.

2) Tail lengths $(n=9)$

$\{3.5, 3.7, 4.5, 3.9, 3.9, 4.1, 4.3, 4.6, 3.8\}$

(a) A point estimate of population mean is the sample mean:

$$\bar{x} = \frac{\sum x_i}{n} = 4.03$$

(b) A $(1-\alpha)$ confidence interval for the mean μ is: $1-\alpha = .94 \Rightarrow \alpha = .06$

$$\bar{x} - t_{\alpha/2, n-1} \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$$

$\bar{x} = 4.03$; $s = \sqrt{\sum \frac{(\bar{x}-x_i)^2}{n-1}} \approx .37$

$n=9$, $t_{\alpha/2, n-1} = t_{0.03, 8} = qt(1-0.03, 8)$

≈ 2.19

$$\Rightarrow 4.03 - 2.19 \frac{.37}{\sqrt{9}} < \mu < 4.03 + 2.19 \frac{.37}{\sqrt{9}}$$

$\Rightarrow 3.76 < \mu < 4.29$ (94% CI)

(c) A 99% CI $\Rightarrow 1-\alpha = .99 \Rightarrow \alpha = .01$

$t_{\alpha/2, n-1} = t_{0.005, 8} = qt(1-0.005, 8)$

≈ 3.36

$\Rightarrow 4.03 - 3.36 \frac{.37}{3} < \mu < 4.03 + 3.36 \frac{.37}{3}$

$\Rightarrow 3.61 < \mu < 4.44$ (99% CI)

3) $n = 64$ Time to make circuit board

$\bar{x} = 15.5$ and $s = 3.2$ since n large take $\sigma \approx s$

$(1-\alpha)$ CI in this case takes the form

$$\bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad (\sigma \approx s)$$

$1-\alpha = .93 \Rightarrow \alpha = .07$

$z_{\alpha/2} = qnorm(1 - .07/2) = 1.81$

$$\Rightarrow 15.5 - 1.81\,\frac{3.2}{8} < m < 15.5 + 1.81\,\frac{3.2}{8}$$

$$\Rightarrow 14.78 < m < 16.22 \qquad (93\%\ CI)$$

If the estimate is 16 min, the max error occurs if m is close to 14.78

so $|\text{max error}| < 16 - 14.78 = 1.22$ min

4) $n = 99$ scores

$s^2 = 127$ (sample variance)

Form a 98% CI for $\sigma \Rightarrow \alpha = .02$

$$\Rightarrow \frac{(n-1)s^2}{\chi^2_{\alpha/2,\,n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,\,n-1}}$$

$\chi^2_{.01,98} \approx 133.47 \approx$ qchisq(1-0.01, df=98)

$\chi^2_{.99,98} \approx 68.39 \approx$ qchisq(0.01, df=98)

$$\Rightarrow \frac{99(127)}{133.47} < \sigma^2 < \frac{99(127)}{68.39}$$

$$\Rightarrow 94.20 < \sigma^2 < 183.84$$

$$\Rightarrow 9.70 < \sigma < 13.55 \quad (98\% \text{ CI for } \sigma)$$

5) Two populations

$n_1 = 8$; $\bar{X}_1 = 72.7$ and $s_1 = 6.35$ book

$n_2 = 12$; $\bar{X}_2 = 76.9$ and $s_2 = 4.85$ online

$\alpha = 0.1 \Rightarrow \alpha/2 = 0.05$

90% CI: $(\bar{X}_2 - \bar{X}_1) \pm t_{\alpha/2, n_1+n_2-2} \sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$

$$= (76.9 - 72.7) \pm t_{.05,18} \sqrt{33.78\left(\frac{1}{8} + \frac{1}{12}\right.}$$

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} \approx 30.06$$

$$t_{.05,18} = qt(1-0.05, 12+18-2) \approx 1.70$$

$$\Rightarrow -0.129 \leq \mu_2 - \mu_1 \leq 8.53 \quad (.90 \text{ CI})$$

To test the claim that online course yields mean test score at least 1pt higher, set up the test:

$H_0: \mu_2 - \mu_1 = 0.99$ to 2 decimal places

$H_1: \mu_2 - \mu_1 > 0.99$ use $\alpha = .01$

decision rule: reject H_0 if $t_s > t_{\alpha, n_1+n_2-2}$

$$t_{\alpha, n_1+n_2-2} = t_{0.01, 18} = qt(1-0.01, 18) \approx 2.55$$

$$t_s = \frac{\bar{x}_2 - \bar{x}_1 - d}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{(76.9 - 72.7) - 0.99}{\sqrt{30.06\left(\frac{1}{8} + \frac{1}{12}\right)}}$$

$$\approx \frac{3.21}{2.502} = 1.28$$

Interestingly, $t_s < t_{\alpha, n_1+n_2-2}$ so we <u>cannot</u> reject H_0. This is due to the fact that s_1, s_2 are both on the order of $\bar{x}_2 - \bar{x}_1$ difference.

6) population normal

$H_0: \mu = 1$ $n = 9$

$H_1: \mu = 2$ $\sigma^2 = .25$

Reject Ho when $\bar{x} > 2$ or $\bar{x} < .6$

(a)

$\alpha = P(\text{type I error}) = P(\text{reject Ho when Ho true})$

$= P(\bar{X} > 2 \text{ or } \bar{X} < 0.6 \mid \mu = 1)$ mutually exclusive

$= P(\bar{X} > 2 \mid \mu = 1) + P(\bar{X} < 0.6 \mid \mu = 1)$

$= P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} > \frac{2 - \mu}{\sigma/\sqrt{n}} \right) + P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{0.6 - \mu}{\sigma/\sqrt{n}} \right)$

$= P\left(Z > \frac{1}{\sqrt{.25}/3} \right) + P\left(Z < \frac{-.4}{\sqrt{.25}/3} \right)$

$= P(Z > 6) + P(Z < -2.4)$

$= (1 - P(Z \leq 6)) + P(Z \leq -2.4)$ Z continuous std. normal r.v.

$= 1 - \text{pnorm}(6) + \text{pnorm}(-2.4)$

≈ 0.0082

(b) $\beta = P(\text{type II error})$

$= P(\text{accept } H_0 \text{ when } H_0 \text{ is false})$

$= P(\text{accept } H_0 \text{ when } H_1 \text{ is true})$

$= P(.6 \le \bar{X} \le 2 \mid \mu = 2)$

$$= P\left(\frac{.6-\mu}{\sigma/\sqrt{n}} \le Z \le \frac{2-\mu}{\sigma/\sqrt{n}} \right)$$

$= P(-8.4 \le Z \le 0)$

$= P(0 \le Z \le 8.4) = P(Z \le 8.4) - 0.5$

$= \text{pnorm}(8.4) - 0.5 \approx 0.5$

(c) power of test

$= 1 - \beta = 1 - 0.5 = 0.5$