1) Recall that for $|r|<1$, the geometric series with inf. many terms has the sum:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

The partial sum of n terms:

$$S_n = a + ar + \dots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$$

$$\sum_{i=4}^{\infty} \left(\frac{1}{2}\right)^i = \underbrace{\left(\frac{1}{2}\right)^4}_{a} + \left(\frac{1}{2}\right)^5 + \dots$$

$a = \left(\frac{1}{2}\right)^4$ and $r = \frac{1}{2}$

$$\Rightarrow \text{sum} = \frac{a}{1-r} = \frac{\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

The finite sum:

$$\sum_{i=10}^{100} \left(\frac{1}{4}\right)^i = a\left(\frac{1-r^n}{1-r}\right) \quad \text{with}$$

$a = \left(\frac{1}{4}\right)^{10}$; $r = \frac{1}{4}$; $n = 100 - 10 + 1 = 91$
(# of terms)

$$\Rightarrow \text{sum} = \left(\frac{1}{4}\right)^{10}\left(\frac{1-\left(\frac{1}{4}\right)^{91}}{1-\frac{1}{4}}\right) \simeq 10^{-6}$$

2) A geometric random variable measures the # of Bernoulli trials needed to get a first success. Note each trial has either success or failure as outcome and the trials are independent of one another.

The possible values are $1, 2, 3, \ldots$

(ex, you can keep flipping coins and theoretically not get a head even after a large number of trials).

PDF: probability density function

$$f_p^g(x) = q^{x-1}p = (1-p)^{x-1}p = P(X_g = x)$$

where $X_g \sim Geo(p)$ (from Geometric distribution with parameter p).

CDF : cumulative distribution function

$$F(x) = P(X_g \le x) = P(X_g \le \lfloor x \rfloor)$$

where $\lfloor x \rfloor = \text{floor}(x)$

note: geo var values start at 1.

$$P(X_g \le \lfloor x \rfloor) = \sum_{k=1}^{\lfloor x \rfloor} f_p^g(k)$$

$$= \sum_{k=1}^{\lfloor x \rfloor} pq^{k-1} = p \sum_{k=1}^{\lfloor x \rfloor} q^{k-1}$$

$$= p\,a\left[\frac{1-r^n}{1-r}\right] = p\left[\frac{1-q^{\lfloor x \rfloor}}{1-q}\right] =$$

$$= p\left[\frac{1-q^{\lfloor x \rfloor}}{p}\right] = 1-q^{\lfloor x \rfloor}$$

(note r = ratio between two terms $= q$,

n = # of terms $= \lfloor x \rfloor - 1 + 1 = \lfloor x \rfloor$,

a = first term $= q^0 = 1$)

When x is an integer, $\lfloor x \rfloor = x$.

$$P(X_g \ge 12) = 1 - P(X_g < 12) = 1 - P(X_g \le 11)$$

$\Rightarrow P(X_g \geq 12) = 1 - F(11)$ where $F = F_p^g$ is the CDF function. Notice that the above equalities follow from the fact that $X_g \sim Geo(p)$ can take on only integral values $1, 2, 3, \ldots$

$$\Rightarrow P(X_g \geq 12) = 1 - (1 - q^{11}) = q^{11} = (1-0.1)^{11} = 0.9^{11}$$

3) Baye's rule:

Suppose $S = B_1 \cup B_2 \cup \ldots \cup B_K$; B_j disjoint $1 \leq j \leq K$

Then,

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(B_j)P(A|B_j)}{\sum_{j=1}^{K} P(B_j)P(A|B_j)}$$

where we have used the total probability formula for $P(A)$. The significance is that it allows us to evaluate the probability of event B_j given A using the probabilities of A given B_j (which maybe easier to evaluate).

3) Define the events,

A: that from 5 fish drawn one by one with replacement from a certain tank, 3 are gold and 2 are blue.

B_i : tank i was used $(i=1,2,3)$

We need $P(B_i | A)$ for all i. We apply Baye's thm:

$$P(B_i | A) = \frac{P(B_i) P(A | B_i)}{\sum_{i=1}^{3} P(B_i) P(A | B_i)}$$

Note that $P(B_i) = \frac{1}{3}$ for each i. (equal probability of choosing either tank).

$P(A|B_i)$ is the probability of drawing fish as in A having used tank i.

Tank 1: twice as many blue fish as gold fish.
Define success as drawing a gold fish (for each tank)
what is p?

Suppose tank 1 has n fish. x = amt of gold fish.

Then $2x + x = n = 3x \Rightarrow x = \frac{n}{3}$

$y = \frac{2}{3}n$ = amt of blue fish.

$\Rightarrow p = \frac{x}{n} = \frac{1}{3}$ (probability of choosing gold fish from tank 1)

$$P(A|B_1) = \underbrace{\binom{5}{3}}_{\text{3 gold "successes" out of 5 trials}}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 \approx 0.165$$ (prob of choosing 3 gold out of 5 from tank 1)

note that we have 5 Bernoulli trials (either blue or gold fish)

Tank 2: 3 gold and 10 blue $\Rightarrow p = \frac{3}{13}$

$$P(A|B_2) = \binom{5}{3}\left(\frac{3}{13}\right)^3\left(\frac{10}{13}\right)^2 \approx 0.073$$

Tank 3: four times as many gold fish as blue fish.

$x + 4x = m$ (x = amt blue) $\Rightarrow 5x = m \Rightarrow x = \frac{m}{5} \Rightarrow q = \frac{1}{5}, p = \frac{4}{5}$

$$P(A|B_3) = \binom{5}{3}\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^2 \approx .2048$$

$$\Rightarrow P(B_1|A) = \frac{P(B_1)P(A|B_1)}{\sum P(B_i)P(A|B_i)} \approx \frac{\frac{1}{3}(0.165)}{\frac{1}{3}(0.165) + \frac{1}{3}(0.073) + \frac{1}{3}(.2048)}$$

$\approx .372$

$$P(B_2|A) = \frac{P(B_2)P(A|B_2)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} \approx \frac{\frac{1}{3}(0.073)}{.148} \approx .165$$

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} \approx \frac{\frac{1}{3}(0.2048)}{.148} \approx .462$$

Note that $\sum P(B_i|A) = 1$.

5) Here we use the law of total probability.

Since all registered voters belong to party I, II, or III:

Let I: event that registered voter belongs to party I

II: " " belongs to party II

III: " " belongs to party III.

$P(I) = .10$; $P(II) = 0.70$; $P(III) = .20$

S: registered voter supports proposal

$P(S|I) = .10$; $P(S|II) = .85$; $P(S|III) = .50$

$$\Rightarrow P(S) = \sum_{i} P(\text{belongs to } i)\, P(S|\text{ belongs to } i)$$

$= P(\mathrm{I})P(S|\mathrm{I}) + P(\mathrm{II})P(S|\mathrm{II}) + P(\mathrm{III})P(S|\mathrm{III})$

$= 0.10(0.10) + 0.70(0.85) + 0.20(0.50)$

$= 0.705$

$\approx 70\%$ chance of registered voter supporting the new law.

6) The no-memory property of the Geometric distribution states that:

$$P(X>(j+k)\,|\,X>j) = P(X>k)$$

Probability that k more trials needed for success from current point independent of the fact that one knows j trials have already occurred without success.

$$P(X \leq k) = F(k) \quad (\text{CDF function})$$

$$P(X>k) = 1 - P(X \leq k) = 1 - F(k) = 1-(1-q^k)$$
$$= q^k$$

note that κ is an integer so $\lfloor \kappa \rfloor = \kappa$.

$$P(X > (j+\kappa)) = \varepsilon^{j+\kappa}; \quad P(X > j) = \varepsilon^{j}$$

$$\Rightarrow P(X > (j+\kappa) \mid X > j) = \frac{P((X > (j+\kappa)) \cap (X > j))}{P(X > j)}$$

$$= \frac{P(X > j+\kappa)}{P(X > j)} = \frac{\varepsilon^{j+\kappa}}{\varepsilon^{j}} = \varepsilon^{\kappa}$$

$$= P(X > \kappa)$$

Note that $[X > (j+\kappa)] \cap [X > j] = X > (j+\kappa)$.

7) 8w, 6m, 4b, 5g

(a) select 2m, 4w, 3b, 3g:

$$\binom{6}{2} \times \binom{8}{4} \times \binom{4}{3} \times \binom{5}{3} = 42{,}000$$

This is the amount of possible selections without restrictions (order here is <u>not</u> relevant).

(b) A particular m and w must be selected. Then we have 5m left and and 7w left and we choose 1m from 5m and 3w from 7w. Thus, with this restriction the # of selections is:

$$\binom{5}{1} \times \binom{7}{3} \times \binom{4}{3} \times \binom{5}{3} = 7000$$

order of selection does not matter.

8) 10 books: 6 math, 4 history

All books are distinct. Here the order matters. We want 3 of the 10 books to be together (doesn't matter they are math books, since all books are distinct).

$$P(\text{3 books together}) = \frac{\text{\# of arrangements with 3 of the books together}}{\text{\# of arrangements with no restrictions}}$$

denominator = $10!$

Suppose 3 books are one unit. Then we have

$$\begin{array}{cccccccc} \boxed{\text{3 books}} & \square & \square & \square & \square & \square & \square & \square \\ 1\text{-}3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$8!$ ways to arrange these

$\boxed{\text{3 books}} = \square\square\square \Rightarrow 3!$ ways to arrange these amongst 3 spots.

$$\Rightarrow P(\text{3 books together}) = \frac{8!\,3!}{10!}$$

$$= \frac{1}{15}$$

9) X_b : Binomial r.v. denoting # of people having bad reaction to a vaccine.

$$P(X_b > 3) = 1 - P(X_b \leq 3) = 1 - \sum_{k=0}^{3} P(X_b = k)$$

$$= 1 - \sum_{k=0}^{3} \binom{1000}{k} (0.001)^k (0.999)^{1-k} \simeq 0.019$$

Using Poisson distribution, we can approximate to high accuracy since n large and p small. (rare event). Let X_p be Poisson r.v. counting # bad reactions.

$$\Rightarrow P(X_p > 3) = 1 - P(X_p \leq 3) = 1 - \sum_{k=0}^{3} P(X_p = k)$$

$$= 1 - e^{-\lambda} \sum_{k=0}^{3} \frac{\lambda^i}{i!}$$

where $\lambda = np = 1000(0.001) = 1$

$$= 1 - \frac{1}{e}\left[1 + 1 + \frac{1}{2} + \frac{1}{6}\right]$$

≈ 0.0189

Note that the calculation is significantly easier.

10) $E[X_b] = np = 1$; $Var[X_b] = npq = 0.999$

$E[X_g] = \lambda = 1$; $Var[X_g] = \lambda = 1$.

note $\sigma(X_b) = \sqrt{npq}$; $\sigma(X_g) = \sqrt{\lambda}$

HW7, part 2 $\quad n = 50, \quad \bar{x} = 5, \quad s = .75$

Use s (sample std deviation) as approximation to σ (population std deviation).

By CLT, since $n \geq 30$, $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ (approximately)

want to calculate $P(|\bar{x} - \mu| < TOL)$

that is the probability that $\bar{x}$ approximates population mean μ within certain tolerance.

$$P(|\bar{x} - \mu| < .8) = P(-.8 < \bar{x} - \mu < .8)$$

$$= P\left(\frac{-.8}{\sigma_{\bar{x}}} < \underbrace{\frac{\bar{x} - \mu}{\sigma_{\bar{x}}}}_{z} < \frac{.8}{\sigma_{\bar{x}}}\right) \quad \text{where } \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \approx \frac{s}{\sqrt{n}}$$

$$\sigma_{\bar{x}} \approx \frac{.75}{\sqrt{50}} = 0.106$$

$$\Rightarrow P(|\bar{x} - \mu| < .8) = P(-7.55 < z < 7.55) \approx 1$$

We are almost certain to approximate within this tolerance.

$$P(|\bar{x} - \mu| < .2) = P(-1.88 < z < 1.88) = 2P(0 < z < 1.88)$$
$$\approx 2(.4699) \approx .94$$

The probability is still high, but has decreased.

Ex) (permutations)

In how many ways can 7 people be seated at a round table, pulling up their own chairs if:

(a) they can sit anywhere

Let 1 be seated anywhere. Then fold out circle:

$$\underset{1}{\square}\,\underset{2}{\square}\,\underset{3}{\square}\,\underset{4}{\square}\;\boxed{1}\;\underset{5}{\square}\,\underset{6}{\square}$$

the remaining 6 people can be seated in

$6! = 720$ (notice, not $7!$)

(b) 2 particular people cannot sit next to each other.

find # of ways they can sit next to each other

$$\underbrace{\boxed{1}\boxed{2}}$$

$2!$ arrangements, 5 remaining people: $5!$ arrangements

# of ways of arranging 7 people with 2 particular people sitting together $= 2!5! = 240$

ways "not together" $= 720 - 240 = 480$

Ex) Five red balls, two white balls, and three blue balls are arranged in a row. If all balls of same color are not distinguishable, how many different arrangements are possible?

$$(5!\,2!\,3!)\underbrace{N}_{\text{\# of arrangements}} = 10!$$

10! # of arrangements if all were different.

$$N = \frac{10!}{5!\,2!\,3!}$$

Ex) From 7 consonants and 5 vowels, how many 7 letter words can be formed consisting of 4 different consonants and 3 different vowels?

$$\underbrace{({}_7C_4)}_{\substack{4\text{ c.}\\ \text{of } 7}} \times \underbrace{({}_5C_3)}_{\substack{3\text{ v. of}\\ 5}} \times \underbrace{({}_7P_7)}_{\substack{\text{\# of ways}\\ \text{of arranging}\\ 7\text{ letters}}} = 35 \times 10 \times 5040$$

$$> 10^6$$