Binomial distribution

success or failure ; $P(\text{success}) = p$; $P(\text{failure}) = q$

p and q do not change frome one trial to the next

trials are independent (called Bernoulli trials)

$$P(X=x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= f_{b,p}(x) \text{ pdf function}$$

mean $\quad m = np$

variance $\quad \sigma^2 = npq$

std deviation $\quad \sigma = \sqrt{npq}$

Law of large numbers for Bernoulli trials:
Let X be the random variable giving the # of successes in n Bernoulli trials, so that $\frac{X}{n}$ is the proportion of successes. Then if p is the probability of success and $\varepsilon > 0$:

$$\lim_{n \to \infty} P\left(\left| \frac{X}{n} - p \right| \geq \varepsilon \right) = 0$$

In the long run, the proportion of successes $\frac{X}{n}$ will be as close as you like to the probability of success in a single trial.

Ex) Toss a fair coin three times. Find the probability that there will be
(a) 3 heads (b) 2 tails, 1 head (c) at least 1 head
(d) not more than 1 tail. X Bernoulli r.v. counts # heads in n trials.

In each case # trials $= n = 3$. define success differently for each part.

(a) $p(3\text{ heads}) = \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{1}{8}$

Notice: $p = \frac{1}{2}, q = \frac{1}{2}$. 3 successes in 3 tosses.
success defined as getting a head

(b) $P(2\text{ tails and }1\text{ head}) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)$

success defined as getting a tail.

$= \frac{3!}{2!\,1!}\,\frac{1}{4\cdot 2} = \frac{3}{8}$

if opposite then, success is a head and result is $\binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2$ yields same result.

(c) $P(\text{at least one head}) =$

$= P(1\text{ head}) + P(2\text{ heads}) + P(3\text{ heads})$

$= 1 - P(\text{no heads})$

success defined as having a head

$= 1 - \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = \frac{7}{8}$

(d) $P(\text{not more than one tail}) = P(0\text{ tails or }1\text{ tail})$
$= P(0\text{ tails}) + P(1\text{ tail}) = \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 + \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)$
defined success as getting a head.

$$= \left(\frac{1}{2}\right)^3 + \frac{3}{2}\,\frac{1}{4\cdot 2} = \frac{1}{2\cdot 2\cdot 2} + \frac{3!}{2!\,1!}\,\frac{1}{2\cdot 2\cdot 2}$$

$$= \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$$

Ex) Suppose a fair die is tossed five times.

Find the probability that a 3 will appear

(a) twice (b) at most once (c) at least two times

X counts # of times 3 occurs.

$n = \#$ of trials $= 5$;

(a) $P(3 \text{ occurs twice}) = P(X=2) = \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$

two successes in 5 trials ; 3 failures

(b) $P(3 \text{ occurs at most once}) = P(X \le 1)$

$$= P(X=0) + P(X=1) = \binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 + \binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4$$

success : get a 3

no 3's out of 5

one 3 out of 5

$$= \frac{3125}{3888}$$

(c) $P(3 \text{ occurs at least } 2 \text{ times})$

$$= P(X \ge 2) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$$

$$= 1 - P(3 \text{ occurs at most once}) = 1 - P(X \le 1)$$

$$= 1 - [P(X=0) + P(X=1)] = \frac{763}{3888}$$