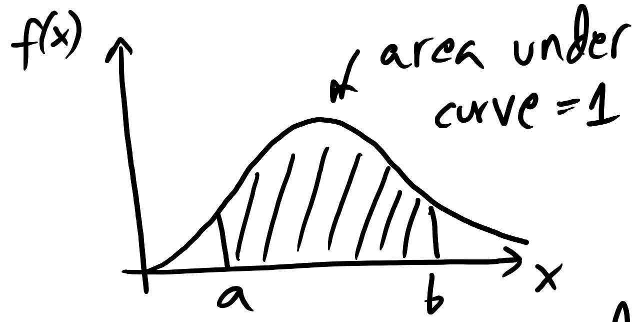
Continuous Probability Distributions



The area under the curve between two values a and b gives the probability that a random variable having this continuous distribution will take on a value $x \in (a,b)$ (on an interval from a to b). $P(a < x < b) = P(a \leq x \leq b)$.

Note: main difference from discrete distributions is that the number of possible values of a random variable X is not countable:

Ex) discrete, $X = 0, 1, 2, 3, \ldots$ countable (though possibly infinite # of values)

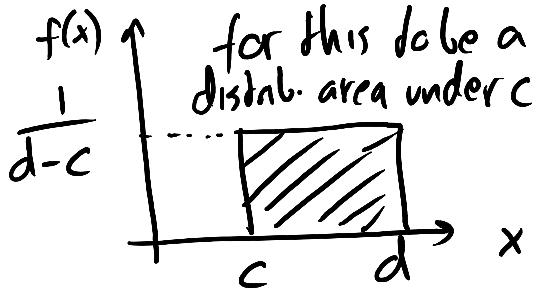
Ex) continuous, $X \in (0, 4) \rightarrow$ $x = 1.21335$, $x = 1.2133573$ etc

uncountable # of values

The Uniform Distribution

Continuous random variables that have equally likely outcomes over an interval $c \leq x \leq d$ possess a uniform probability distribution.

Total area of rectangle = base × height $= (d-c)\left(\frac{1}{d-c}\right) = 1$

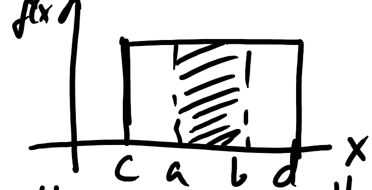


for this to be a probability distrib. area under curve = 1.

f(x) is called the probability density function

$$f(x) = \frac{1}{d-c} \; ; (c \le x \le d)$$ prob. density function for a uniform distribution

$$\mu = \frac{c+d}{2} \; ; \; \sigma = \frac{d-c}{\sqrt{12}}$$



$$P(a < x < b) = \frac{b-a}{d-c}$$ equal to the area of the rectangle over the interval.

Ex) A bad car dealer sells a used car to an unsuspecting buyer, even though the dealer knows the car will have a breakdown within the next 6 months. The dealer provides a warranty of 45 days on all cars sold. Assume x is uniform dist. r.v.

(a) find mean and std deviation of x.

value of x indicates when car will break down.

$$\mu = \frac{c+d}{2} = \frac{0+6}{2} = 3 \text{ months}$$

$$\sigma = \frac{d-c}{\sqrt{12}} = \frac{6-0}{\sqrt{12}} = \frac{6}{\sqrt{4 \cdot 3}} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

$\simeq 1.73$ months

The uniform probability distribution is:

$$f(x) = \frac{1}{d-c} = \frac{1}{6-0} = \frac{1}{6} \quad (0 \le x \le 6)$$

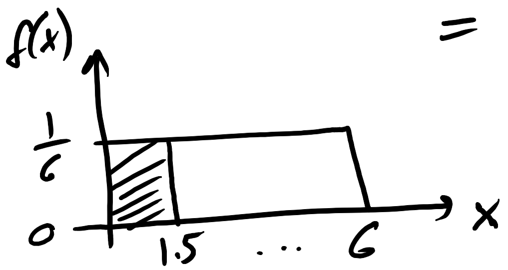
(b) calculate the probability that a breakdown occurs while the car is still under warranty.

45 days $= 1.5$ months
need area under the probability density function $f(x)$ between the points $x=0$ and $x=1.5$.

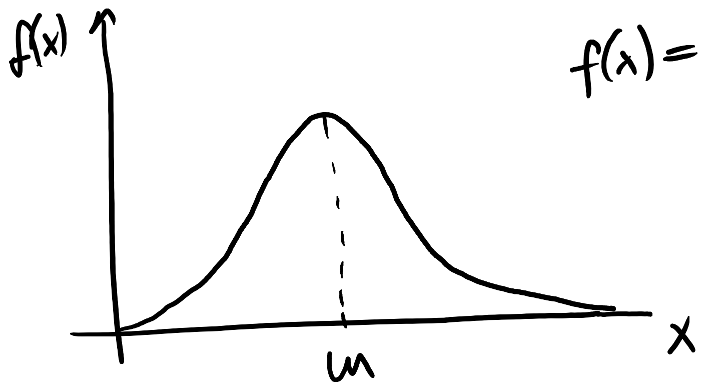
$$\Rightarrow P(0 < x < 1.5) = (\text{base}) \times (\text{height}) = 1.5\left(\frac{1}{6}\right) = 0.25$$

$$= \int_0^{1.5} f(x)\,dx$$

relation to integrals



The Normal Distribution



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{(x-m)}{\sigma}\right]^2}$$

$m = \text{mean}(x)$
$\sigma = \text{std dev}(x)$
$\pi = 3.1416\ldots$
$e = 2.71828\ldots$

$$P(a < x < b) = \int_a^b f(x)\,dx \quad \text{(area under the curve)}$$

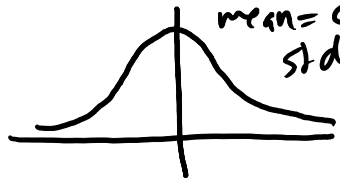
The standard normal random variable is defined by:

$$z = \frac{x-m}{\sigma}$$

where x is a normal random variable with mean m and std deviation σ.

z has mean zero and std deviation 1.

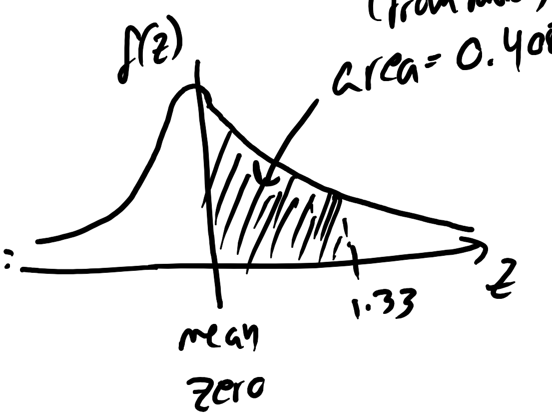
$\Rightarrow$ We generally convert all normal random variables and then use a table (or software) to find probabilities.



Ex) Suppose we know that a length of time between charges of a pocket calculator has a normal distribution with mean 50 hours and std deviation of 15 hours. What is the probability that a calculator will be charged between 50 and 70 hours?

① convert to standard normal random variable

$$z = \frac{x - \mu}{\sigma} = \frac{x - 50}{15}$$



② plug in particular $x = 70$:

$$z = \frac{70 - 50}{15} = \frac{20}{15} \simeq 1.33$$

The measurement 70 is 1.3 std deviations above the mean of 50.

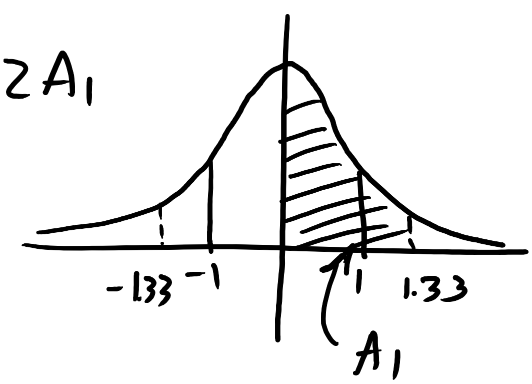
③ Refer to the table. Notice that z-scores are listed in the left hand column of the table. To find the area corresponding to z score 1.33, we first locate the value 1.3 in the left hand column. Since this column lists z-values to one decimal place only, we refer to the top row of the table to get the second decimal place, .03.

Finally, we locate the number where the row labeled $z=1.3$ and the column labeled .03 meet. This number represents the area between the mean m and the measurement with z-score 1.33.

$\Rightarrow A = .4082$ = probability that the calculator operates between 50 and 70 hrs before charging.

Ex) various probabilities of standard normal random variables:

$P(-1.33 < z < 1.33) = 2A_1$



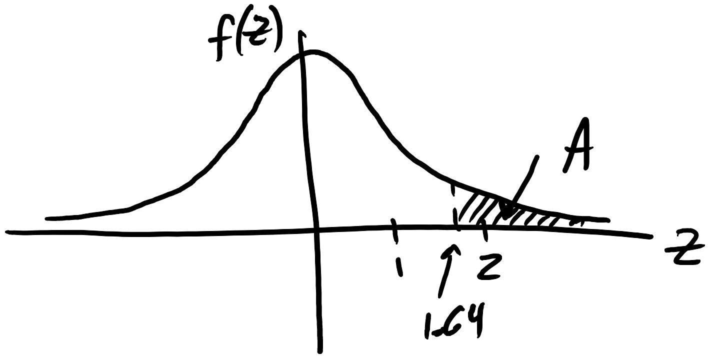
$P(-1.33 < z < 1.33) = P(-1.33 < z < 0) + P(0 \le z < 1.33)$

mutually exclusive events

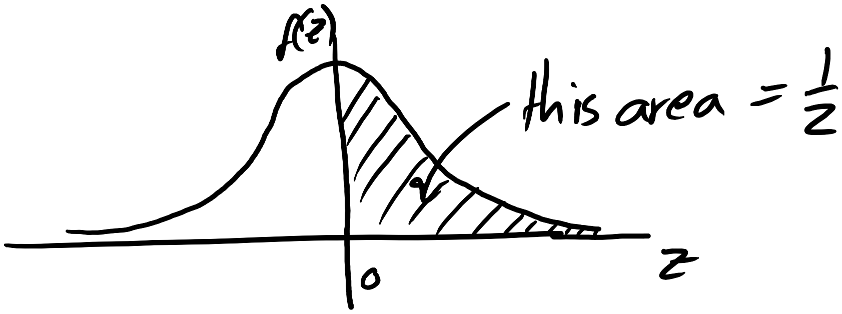
$= 2A_1 = 2(.4082) = .8164$

Ex) Find the probability that a standard normal random variable exceeds 1.64.

$\Rightarrow$ Find $P(z > 1.64)$

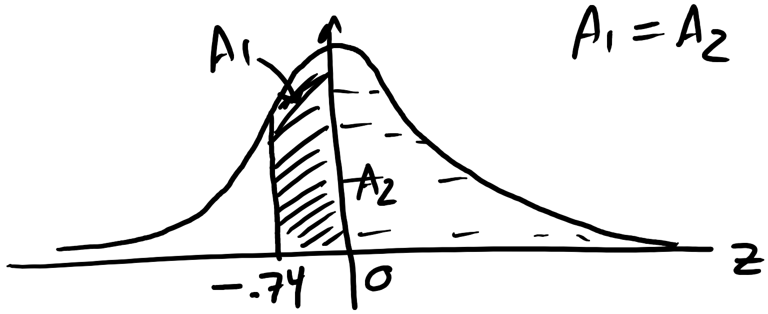


Note that the std normal distribution is symmetric about its mean and that total area under curve is 1.



$$P(z > 1.64) = \frac{1}{2} - P(0 < z < 1.64) = .5 - .4495 = .0505$$

Ex) Find the probability that a normal random variable lies to the right of a point $-.74$ standard deviations from its mean.



$P(z > -.74) = A_1 + A_2 = .2704 + .5 = .7704$

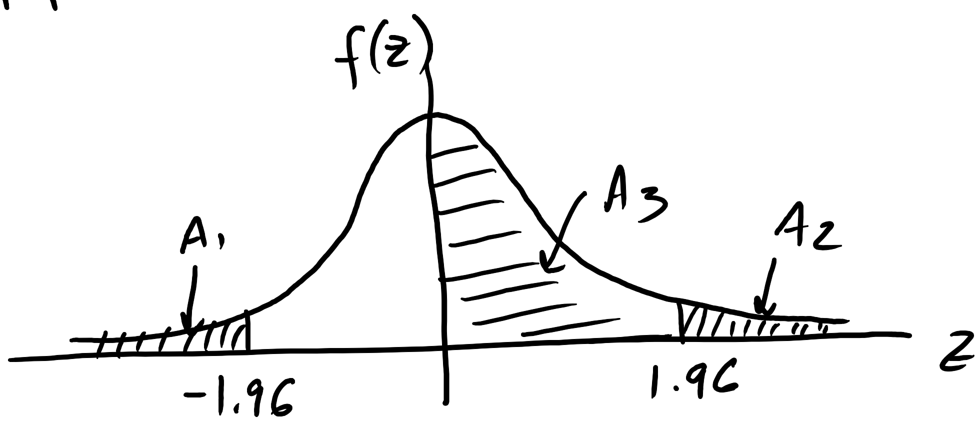
notice that to get A_1 note that

$P(-.74 < z < 0) = P(0 < z < .74)$

and use the table to get $.2704$.

Ex) Find the probability that a normal random variable lies more than 1.96 std deviations from its mean in either direction.

$P(|z| > |1.96|) = P(z < -1.96 \text{ or } z > 1.96)$



$P(z > |1.96|) = A_1 + A_2 = .0250 \times 2 = 0.05$

where $.0250 = .5 - \underbrace{.4750}_{A_3}$

the standard normal distribution (further look)

Density function:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\left.\begin{array}{l} Pr(z>0) = Pr(z<0) \\ Pr(z>0) + Pr(z<0) = 1 \end{array}\right\} \quad z = \frac{x-\mu}{\sigma}$$

std normal r.v.

Ex) Find $Pr(-.47 < z < .94)$

$Pr(\cdots) = A(-.47) + A(.94) = A(.47) + A(.94)$

$A(.47) = Pr(z < .47) = .18082$ from table

$A(.94) = Pr(z < .94) = .32639$ from table

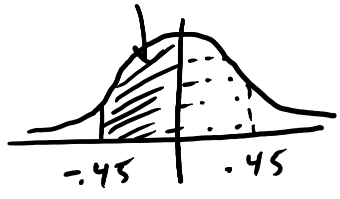
$Pr(\cdots) = .18082 + .32639 = .50721$

cumulative probability function

$\Phi(y) = Pr(z \leq y)$ where z is std normal r.v.

Ex) Find $\Phi(-.45)$

$A(-.45) = A(.45) = .17364$ from table

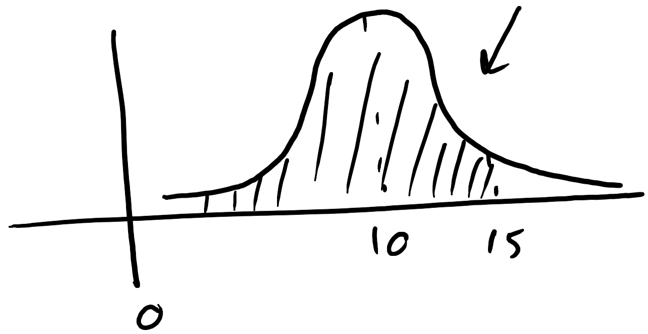


$\Rightarrow \Phi(-.45) = \Phi(0) - A(-.45)$

$= .5000 - .17364 = .32636$

Ex) Given that random variable X has a normal distribution with mean 10 and std. deviation 4, find $P(X < 15)$:

not the standard normal curve

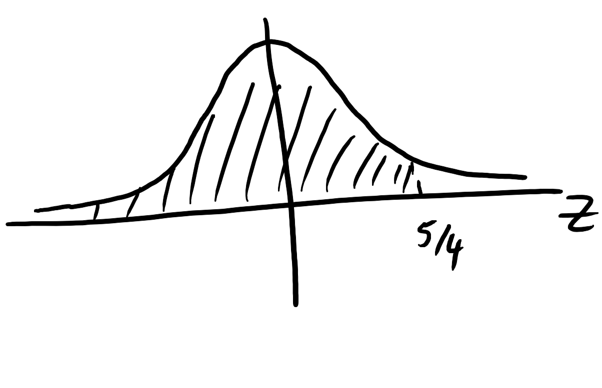


$Z = \dfrac{X-10}{4}$ is standard normal

$X=15$ corresponds to $z = \dfrac{15-10}{4} = \dfrac{5}{4}$

So $Pr(X<15) = Pr(z < \frac{5}{4})$ since

$Pr(X<15) = Pr\left(\dfrac{X-10}{4} < \dfrac{15-10}{4}\right)$



$Pr(z < \frac{5}{4}) = \Phi(\frac{5}{4})$

$= .5000 + A(1.25)$

$= .5000 + .39439$

$= .89439$

Ex) Given a normal population with mean $\mu = 25$ and $\sigma = 5$, find the probability that an assumed value of the variable will fall in the interval 20 to 30.

$$Pr(20 < x < 30) = Pr\left(\frac{20-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{30-\mu}{\sigma}\right)$$

$$= Pr\left(\frac{20-25}{5} < z < \frac{30-25}{5}\right) = Pr(-1 < z < 1).$$

$$Pr(0 > z > -1) = Pr(0 < z < 1) = .341$$

$$Pr(-1 < z < 1) = Pr(0 > z > -1) + Pr(0 < z < 1)$$

$$= 2(.341) = .682$$