Combinations / Permutations / Counting

<u>Multiplication</u> If a choice consists of two steps, of which the first can be made in m ways and the second in n ways, then the whole choice can be made in $m \cdot n$ ways.

Ex) travel agency offers weekend trips to 12 different cities by air, rail, or bus.

Can arrange trip in $12 \times 3 = 36$ ways.

Multiplication of choices (generalized)

If a choice consists of k steps, of which the first can be made in n_1 ways, the second in n_2 ways and the k-th in n_k ways then the whole choice can be made in:

$$n_1 \cdot n_2 \cdots n_k \text{ ways}$$

Ex) A car dealer offers a car in 4 body styles, ten colors, and with a choice of 3 engines.

$4 \cdot 10 \cdot 3 = 120$ ways in which a person can order a car.

Permutations

A set of objects in which position (order) is important. HHT different from HTH.
two heads one tail for both, but different order.

The number of permutations of r objects selected from a set of n distinct objects is

$$nPr = n(n-1)(n-2)\cdots(n-r+1)$$

$$= \frac{n!}{(n-r)!}$$ factorial notation

$1! = 1$ $\quad$ $2! = 2 \cdot 1 = 2$ $\quad$ $3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ $\quad$ $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ $\quad$ Also, $\underline{0! = 1}$.

why? first selection made from n objects, second from $(n-1)$ remaining objects, third from $(n-2)$, and the r-th from $\underbrace{n-(r-1)}_{=(n-r+1)}$ remaining objects

$$_nP_r = \frac{n!}{(n-r)!} = \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots(n-n+1)}{(n-r)(n-r-1)\cdots(n-n+1)}$$

$$= n(n-1)\cdots(n-r+1)$$

Ex) Find the number of ways in which four of ten new movies can be ranked $\{1,2,3,4\}$ by a panel of movie critics:

$n=10, r=4 \Rightarrow \quad {}_{10}P_4 = 10\cdot 9\cdot 8\cdot 7 = 5040$

$${}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10\cdot 9\cdot 8\cdot 7$$

Notice that to find the formula for the number of permutations of n distinct objects taken all together, we substitute in $r=n$ to get:

$$_nP_n = n! \qquad \text{since } 0! = 1$$

Ex) In how many ways can 8 teaching assistants be assigned to 8 sections of a course in algebra?

$$_8P_8 = 8! = 40{,}320$$

Ex) Number of permutations of the four letters $\{a,b,c,d\}$ taken two at a time.

$$_4P_2 = \frac{4!}{2!} = 4\cdot 3 = 12$$

We can list all the 12 arrangements:

$$\left\{\begin{matrix} ab & ac & ad & bc & bd & cd \\ ba & ca & da & cb & db & dc \end{matrix}\right\}$$

Notice: there are only six <u>unordered</u> ways to choose two letters from the four, as given by the top raw.

Ex) A club has 100 members. A president, vice pres., secretary and a treasurer are to be selected from the membership. All members are eligible for the offices but each member can hold at most one office.

(a) How many distinct choices of officers are there?

$$_{100}P_4 = 100\cdot 99\cdot 98\cdot 97$$

(b) what is the probability that Lee, Jones, William and Mary (4 particular members) hold all the offices?

Notice: each can occupy one of the offices

${}_4P_4 = 4!$ ways for them to do that (order important)
out of ${}_{100}P_4$ distinct choices

B: event that these four people will hold 4 offices

$$P(B) = \frac{N_B}{N} = \frac{4!}{{}_{100}P_4}$$

Combinations

Order doesn't matter.

The number of ways in which r objects can be selected from a set of n distinct objects is:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$
$$= {}_nC_r$$

Ex) In how many ways can a person choose 3 books from a list of 8 best-sellers?

Notice: order doesn't matter

$B_1 \quad B_2 \quad B_3 \quad \underline{B_4 \quad B_5 \quad B_6} \quad B_7 \quad B_8$

${}_nC_r$ with $n=8$, $r=3$:

$${}_8C_3 = \binom{8}{3} = \frac{8\cdot 7\cdot 6}{3!} = 8\cdot 7 = 56$$

Also, $_8C_3 = \frac{8!}{3!5!} = \frac{8\cdot7\cdot6\cdot5!}{3!\,5!} = \frac{8\cdot7\cdot6}{3!} = 8\cdot7 = 56$

Ex) In how many ways can the director of a research laboratory choose two chemists from among seven applicants and three physicists from among nine applicants.

Two chemists selected in $\binom{7}{2}$ ways.

Three physicists can be selected in $\binom{9}{3}$ ways.

$$\binom{7}{2} = \frac{7!}{2!5!} = \frac{7\cdot6}{2} = 7\cdot3 = 21$$

$$\binom{9}{3} = \frac{9!}{3!\,6!} = \frac{9\cdot8\cdot7}{3\cdot2} = \frac{9\cdot7}{3}\cdot4 = 3\cdot7\cdot4 = 84$$

total # of ways $= \binom{7}{2}\times\binom{9}{3} = 21\times84 = 1764$

When r objects are selected from a set of n distinct objects, n−r of the objects are left. Consequently, there are as many ways of leaving (or selecting) n−r objects from a set of n distinct objects as there are ways of selecting r objects.

$\binom{n}{r} = \binom{n}{n-r}$ for $r = 0, 1, 2, \ldots, n$

Ex) $\binom{75}{72} = \binom{75}{3} = \dfrac{75 \cdot 74 \cdot 73 \cdot 72!}{3!\,72!} = \dfrac{75 \cdot 74 \cdot 73}{3 \cdot 2 \cdot 1}$

$= 67525$

There are as many ways of selecting none of the objects in a set as there are ways of choosing the n objects which are left.

$\binom{n}{0} = \binom{n}{n-0} = \binom{n}{n} = \dfrac{n!}{n!\,0!} = 1$

Ex) If three of twenty tires are defective and four of them are randomly chosen for inspection what is the probability that one of the defective tires will be included?

There are $n = \binom{20}{4} = 4845$ ways of choosing 4 of 20 tires.

The number of "favorable outcomes" is the number of ways in which one of the three defective tires and three of the 17 nondefective tires can be selected.

$s = \binom{3}{1}\binom{17}{3} = 3 \cdot 680 = 2040$

P(one defective tire and three non-defective tires)

$$= \frac{s}{n} = \frac{\binom{3}{1}\binom{17}{3}}{\binom{20}{4}} = \frac{2040}{4845} = \frac{8}{19}$$

Ex) A card is drawn at random from a deck of 52 cards.

P(3 of clubs <u>or</u> six of diamonds)

$$= \frac{1}{52} + \frac{1}{52} = \frac{2}{52} = \frac{1}{26}$$ (mutually exclusive events)

$P(10 \text{ or spade}) = P(10 \cup S)$ <u>not</u> mutually exclusive

can have a 10 of spades

$$= P(10) + P(S) - P(10 \cap S) = \frac{4}{54} + \frac{1}{4} - \frac{1}{52} = \frac{4}{13}$$

Permutations and Combinations

$$nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

(order important, e.g. assigning people to specific positions in a club: president, vicepres, secretary, etc).

$$nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

(order not important, e.g. assigning people to play basketball from a class).

$rP_r = r! = r(r-1)\cdots 1$; note that $0! = 1$

$rC_r = \frac{r!}{r!} = 1$ (if order doesn't matter one way to put r people in r seats).

if order does matter, there are $r!$ permutations of r people in r seats.

$$\binom{n}{r} = \binom{n}{n-r}$$

when r objects are selected from a set of n, n-r are left.