Ex) Suppose that flaws in some wooden pieces used for furniture occur at random with an avg of 1 flaw per $50\,cm^2$. What is the probability that a $4\,cm \times 8\,cm$ small sheet has (a) no flaws, (b) at most one flaw?

Assume the # of flaws per unit area is Poisson distributed. Let X be Poisson r.v. recording # of flaws in $4\times 8\,cm^2$ sheet.

What is λ? Recall $\lambda = E[X]$.

We have $4 \times 8\,cm^2 = 32\,cm^2$ sheet
We expect one flaw per $50\,cm^2$.

$$\Rightarrow E[X] = \frac{32}{50} \text{ flaws} < 1 \text{ flaw}$$

$$\text{Let } \lambda = \frac{32}{50} = \frac{16}{25}$$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X=0) = \frac{e^{-\frac{16}{25}}\left(\frac{16}{25}\right)^0}{0!} = e^{-\frac{16}{25}} \approx 0.53$$

$P(\text{at most one flaw}) = P(X \leq 1) = P(X=0) + P(X=1)$

$X=0, X=1$ are mutually exclusive events

$$P(X \leq 1) = e^{-16/25} + \frac{e^{\frac{-16}{25}} \left(\frac{16}{25}\right)^1}{1!}$$

$$= e^{-0.64} + 0.64 e^{-0.64} \approx 0.86$$

Ex] Suppose that on avg, in every 3 pages of a book there is one typo. If the # of typos on a single page of a book is a Poisson r.v., what is the probability of at least one error on a certain page of a book?

X r.v. (Poisson) counting # of errors on the page we are interested in.

$$E[X] = \frac{1}{3} = \lambda \quad ; \quad P(X=x) = \frac{\left(\frac{1}{3}\right)^x e^{-1/3}}{x!}$$

$$P(X \geq 1) = 1 - P(X=0) = 1 - e^{-1/3} \approx 0.28$$

Geometric random variable

Suppose that a sequence of independent Bernoulli trials (Binomial trials), each with probability of success $0 < p < 1$, are performed.

Let X be the # of trials until the first success occurs. Then X is a discrete random variable called <u>geometric</u>. It's set of possible values is $1, 2, 3, \ldots$ (countably infinite) and

$$P(X=x) = (1-p)^{x-1} p = f_p^g(x)$$

[note, no limit on # of trials prior to first success

$(x-1)$ trials are failures, x-th trial is a success, successive Bernoulli trials are independent.

$$\sum_{x=1}^{\infty} p(x) = \sum_{x=1}^{\infty} (1-p)^{x-1} p = \frac{p}{1-(1-p)} = 1$$

since $\sum_{x=1}^{\infty} (1-p)^{x-1}$ is a geometric series.

Hence, since $0 \le p(x) \le 1$, $p(x)$ is a probability function.

$$E[X] = \sum_{x=1}^{\infty} x f_p^g(x) = \sum_{x=1}^{\infty} x p (1-p)^{x-1} =$$

$$= \frac{p}{1-p} \sum_{x=1}^{\infty} x (1-p)^x = \frac{p}{1-p} \frac{1-p}{[1-(1-p)]} = \frac{1}{p}$$

where we have used $\sum_{x=1}^{\infty} x r^x = \frac{r}{(1-r)^2}$, $|r|<1$.

Can show that:

$$Var[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= \frac{1-p}{p^2} = \frac{1}{p^2} - \frac{1}{p}$$

where $E[X^2] = \sum_{x=1}^{\infty} x^2 p (1-p)^{x-1}$

Ex) From a deck of 52 cards, we draw cards at random until ace is drawn. What is the probability at least 10 draws are needed? Let X be a geometric r.v. We must then "extend" the problem to allow arbitrary many trials, see below.

$$E[X] = \frac{1}{p} = 13 \left(= \frac{4}{52}\right) \Rightarrow p = \frac{1}{13}$$

Thus,

$$P(X=x) = \left(\frac{12}{13}\right)^{x-1}\left(\frac{1}{13}\right), \quad x=1,2,3,\ldots$$

$$P(X \geq 10) = \sum_{n=10}^{\infty}\left(\frac{12}{13}\right)^{n-1}\left(\frac{1}{13}\right) = \left(\frac{12}{13}\right)^{9} \approx 0.49$$

same as probability of no aces in first nine <u>blind</u> draws.

Note that the maximum number of draws needed is at most 48.

$$\sum_{n=10}^{\infty}\left(\frac{12}{13}\right)^{n-1}\left(\frac{1}{13}\right) = \frac{1}{13}\sum_{n=10}^{\infty}\left(\frac{12}{13}\right)^{n-1}$$

$$= \frac{1}{13}\,\frac{(12/13)^9}{1-12/13} = \left(\frac{12}{13}\right)^{9} \approx 0.49$$

Geometric r.v. can be used assuming we combine many decks together and draw cards from an infinite pool.

Recall <u>geometric series</u>:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots \quad \text{converges for } |r|<1.$$

When $|r|<1$, the sum of ∞ many terms is $\dfrac{a}{1-r}$

$$S_n = a + ar + \ldots + ar^{n-1} \quad \text{(partial sum, n terms)}$$

$$rS_n = ar + ar^2 + \ldots + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$\Rightarrow S_n(1-r) = a(1-r^n) \Rightarrow S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

For $n \to \infty$ and $|r| < 1 \Rightarrow S_n \to \frac{a}{1-r}$

Thus, e.g.:

$$\sum_{n=10}^{\infty} \left(\frac{12}{13}\right)^{n-1} = \frac{(12/13)^9 \leftarrow \text{first term } (a)}{1 - 12/13 \leftarrow r}$$

In order for the event $X=x$ to occur, it is necessary to have $x-1$ failures followed by a success:

$$f_p^g(x) = P(X_g = x) = p\,q^{x-1} \quad \text{where } q = 1-p$$

$$\sum_{x=1}^{\infty} f_p^g(x) = p \sum_{x=1}^{\infty} q^{x-1} = p(1+q+q^2+\dots)$$
$$= p\,\frac{1}{1-q} = \frac{p}{p} = 1$$

<u>No memory property</u>

Suppose $X \sim Geo(p)$, then:

$$P[X > j+k \mid X > j] = P[X > k]$$

$$P[X>j+k \mid X>j] = \frac{P[(X>j+k)\cap(X>j)]}{P[X>j]}$$

For this, we need the CDF of the geometric distribution:

$$F_p^g(x) = P(X_g \leq x) = \sum_{i=1}^{x} f_p^g(i) = \sum_{i=1}^{x} pq^{i-1}$$

$$= p\sum_{i=1}^{x} q^{i-1} = p\underbrace{\left[\frac{1-q^x}{1-q}\right]}_{a\frac{1-r^n}{1-r}} = p\left[\frac{1-q^x}{p}\right]$$

$$= 1-q^x$$

Thus,

$$P(X>k) = \overbrace{1-P(X\leq k)}^{1-F_p^g(x)} = 1-(1-q^k) = q^k$$

$$P(X>j+k) = q^{j+k} = (1-p)^{j+k}$$

$$P(X>j) = q^j = (1-p)^j$$

$$\Rightarrow P[X>j+k \mid X>j] = \frac{P[X>j+k]}{P[X>j]}$$

Note $(X>j+k)\cap(X>j) = \{X>j+k\}$

$$P[X > j+k \mid X > j] = \frac{(1-p)^{j+k}}{(1-p)^{j}}$$

$$= (1-p)^{k} = P[X > k]$$

Thus, knowing j trials have passed without a success does not affect the probability of k more trials being required to obtain a success.

Ex) A job interviewer interviews people for a special skills position. Let p, the probability that he/she succeeds in finding the right person equal to 0.20.

(a) Find probability 4 people interviewed before one is selected.

$\Rightarrow$ X geometric r.v. measures # trials before success

$p = 0.20$

$$P(X=4) = q^{3}p = (0.80)^{3}(0.20) \approx 0.10$$

(b) What is the probability that the interviewer must interview more than 6 people before finding a suitable person?

$$P(X>6) = 1 - P(X \leq 6)$$

$$F_p^g(x) = P(X \leq x) = 1 - q^x$$

$$\Rightarrow P(X>6) = 1 - (1 - 0.8^6) = 0.8^6 \approx 0.26$$

(c) How many people do we expect to be interviewed?

$$\mu = E[X] = \frac{1}{p} = \frac{1}{0.20} = 5$$

$$\sigma^2 = Var[X] = \frac{1-p}{p^2} = \frac{0.80}{(0.20)^2} = 20$$

(d) Suppose 10 people have been interviewed and no suitable candidate is found. What is the probability that more than 6 people will need to be interviewed?

$$P(X>16 \mid X>10) = P(X>6) \approx 0.26 \quad \left(\text{using no memory property}\right)$$

Ex) (4-G10) In blackjack a player is dealt 2 different cards from 52 card deck. Find probability of getting a 2 card hand consisting of ace of clubs and ace of spades.

C_1 : get ace of clubs
C_2 : get ace of spades
} any order is fine

$$P(C_1 \cap C_2) = P(C_1)\, P(C_2 | C_1)$$
$$= \frac{1}{52} \frac{1}{51}$$

Since $P(C_2|C_1) = \dfrac{P(C_1 \cap C_2)}{P(C_1)}$

can also get C_2 first then C_1. $(P(C_1|C_2))$

So probability is $2 \frac{1}{52} \frac{1}{51}$.

Review

Discrete random variables can take on finitely or infinitely many values but the set is countable.

<u>Bernoulli trials</u> : Independent, same success probability

e.g. picking cards face down, tossing coins.

Binomial distribution

X counts # of successes in n trials

Two parameters: n and p.

$$f^b_{n,p}(x) = P(X=x) = \binom{n}{x} p^x q^{n-x}$$

Poisson distribution

X counts # of successes.

$\lambda = np \Rightarrow p = \frac{\lambda}{n}$
fixed.

$$f^P_\lambda(x) = \lim_{n \to \infty} f^b_{n,\frac{\lambda}{n}}(x)$$

(can approximate Binomial distribution with large n).

In general, the approximation is good if $p \le 0.1$ and $\lambda = np \le 5$.

Ex) Ten percent of tools turn out to be defective. Find the probability that in a sample of 10 tools chosen at random, exactly two will be defective using Binomial and Poisson distributions.

$$P(X_b = 2) = \binom{10}{2}(0.1)^2(0.9)^8 \approx 0.1937$$

$$P(X_p = x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad ; \lambda = np = 10(0.1) = 1$$

$$\Rightarrow P(X_p = 2) = \frac{(1)^2 e^{-1}}{2!} \approx 0.1839$$ (bit off from above, but close)

CDF : $F^P(x) = P(X_p \le x)$

$$= e^{-\lambda} \sum_{i=0}^{\lfloor x \rfloor} \frac{\lambda^i}{i!}$$

where $\lfloor x \rfloor = \text{floor}(x)$ $(\lfloor 1.9 \rfloor = 1)$

Ex) Let the probability of bad reaction from flu vaccine be 0.001. Determine the probability that out of 2000 individuals more than 2 will suffer bad reaction.

Let X denote # of individuals suffering bad reaction. X is bernoulli distributed.

However, we can use Poisson distribution to approximate.

$$P(X_p > 2) = 1 - P(X_p \leq 2) = 1 - F^P(2)$$

$$= 1 - \sum_{i=0}^{2} e^{-\lambda} \frac{\lambda^i}{i!}$$

$$\lambda = np = (2000)(0.001) = 2$$

$$P(X_p > 2) = 1 - \left[\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} \right]$$

$$= 1 - 5e^{-2} \approx 0.323$$

$$\approx P(X > 2)$$

Geometric distribution

X_g measures # of trials needed to get first success.

$$f_p^g(x) = \varepsilon^{x-1} p = (1-p)^{x-1} p = P(X_g = x)$$

Other distributions: negative binomial, multi-nomial, etc.