Key concepts: additive rule, mutually exclusive, independent events set relations

Additive rule of probability

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$ $P(A \cap B) \neq 0$

events not mutually exclusive

Ex) Hospital records show 12% of patients are admitted for surgical treatment, 16% admitted for obstetrics and 2% receive both obstetrics and surgical treatment. What is the probability that a patient will be admitted for surgery <u>or</u> obstetrics or <u>both</u>?

A: { patient admitted to hospital gets surgery }

B: { patient admitted to hospital gets obstetrics }

$P(A) = 0.12$; $P(B) = 0.16$; $P(A \cap B) = 0.02$

We are interested in:

$P(A \cup B) = P(A) + P(B) - P(\underline{A \cap} B) = 0.12 + 0.16 - 0.02 = 0.26$

MAE: <u>mutually exclusive events</u>, $A \cap B$ contains no simple events and $P(A \cap B) = 0$.

events are mutually exclusive

Ex) Toss two coins. Find probability of getting at least one head.

A: {observe at least one head}

B: {observe exactly one head}

C: {observe exactly two heads}

Then $A = B \cup C$, $P(B \cap C) = 0$

$P(B \cup C) = P(B) + P(C) - 0 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

sample space: $\{HH, HT, TH, TT\}$

Using complement of event A:

$A: \{HH, HT, TH\}$ = (observe at least one head)

A', A^c: {observe no heads} = $\{TT\}$

$P(A^c) = P(TT) = \frac{1}{4}$ (look at sample space)

$P(A) = 1 - P(A^c) = 1 - \frac{1}{4} = \frac{3}{4}$ (using complement rule)

We call two outcomes independent if and only if, the occurance of one does not affect the probability that another will occur.

Ex) tossing two coins. notice: events A and B are independent.

A: {first coin heads up} : B: {second coin heads up}

$P(A) = \frac{1}{2}$; $P(B) = \frac{1}{2}$ $\Rightarrow$ $P(A \cap B) = P(A)P(B) = \frac{1}{4}$

Ex) 10 coin flips. $P(10 \text{ heads}) = (\frac{1}{2})^{10}$ ← very small!

Ex) A high school has 1000 students. The swimming and diving coach seeks team prospects.

Suppose: there are two teams (swim, diving teams)

200 students can swim well enough to make swim team

100 students can dive well enough to make diving team

30 students swim and dive well enough to make both teams.

Find] $P(s) = P(\text{swim fast})$

$P(d) = P(\text{dive well})$

$P(s \cap d) = P(\text{swim fast and dive well})$

$P(s \cup d) = P(\text{swim fast or dive well})$

Consider probabilities that the student will pick someone who:

In this case $P(s \cup d) \gg P(s \cap d)$

$\Rightarrow \quad P(s) = \frac{200}{1000} = 0.2 \quad ; \quad P(d) = \frac{100}{1000} = 0.1$

$P(s \cap d) = \frac{30}{1000} = 0.030$

$P(s \cup d) = P(s) + P(d) - P(s \cap d)$

$= 0.200 + 0.100 - 0.030 = 0.270$

Thus, 270 of 300 students qualify fo either or both teams.

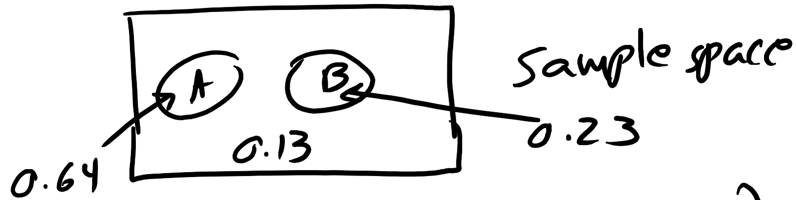
Ex] mutually exclusive events:

A: I am wearing black shoes

B: I am wearing blue sneakers

Suppose $P(A) = 0.64$ and $P(B) = 0.23$

$P(A \cap B) = 0$



$P(A \cup B) =$ P(wear black shoes or blue sneakers) $=$

$= \underline{0.23 + 0.64 = 0.87}$ (sum of probabilities)

$P([A \cup B]') =$ P(donot wear either black shoes or blue sneakers) $=$ $1 - P(A \cup B)$

$= 1 - 0.87 = 0.13$

Relationship between AND and OR

1. The probability of one event OR another OR both is the sum of probabilities of two events minus the probability of one event and another event.

$\underline{P(A \text{ union } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)}$ (addition rule)

2. The probability of one event and another is equal to the sum of probabilities of the two events minus the probability of one event or the other.

$P(A \text{ and } B) = P(A \cap B) = P(A) + P(B) - P(A \cup B)$

Independent events Two events are said to be independent if information about one of them does not help us determine how likely the other event is to occur.

Ex) toss 2 coins.
A: heads on 1st
B: tails on 2nd.

If A and B are independent

$P(A \text{ and } B) = P(A)\,P(B) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$

Ex) Suppose events A, B, C have probabilities: more complex set relations

$\boxed{P(A) = \frac{1}{2}}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$

$A \cap C = \phi$, $B \cap C = \phi$, $\boxed{P(A \cap B) = \frac{1}{6}}$

Find (a) $P([A \cap B]')$ (b) $P(A \cap B')$

(c) $P[(A \cup B)']$ (d) $P(A' \cap B')$

(a) $P([A \cap B]') = 1 - P(A \cap B) = 1 - \frac{1}{6} = \frac{5}{6}$

(b) Use $\boxed{(A \cap B') \cup (A \cap B) = A}$; $\boxed{(A \cap B') \cap (A \cap B) = \phi}$

$\frac{1}{2} = P(A) = P[(A \cap B') \cup (A \cap B)] =$

$= P(A \cap B') + P(A \cap B) - 0^*$

$= P(A \cap B') + \frac{1}{6} \Rightarrow P(A \cap B') = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

* notice that the events $(A \cap B')$ and $(A \cap B)$ are mutually exclusive.

(c) $\boxed{P([A \cup B]') = 1 - P(A \cup B)}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (addition rule)

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

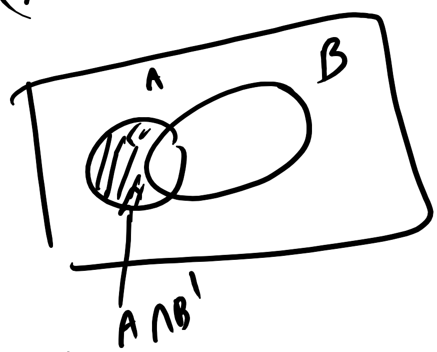
Thus, $P([A \cup B]') = 1 - \frac{2}{3} = \frac{1}{3}$

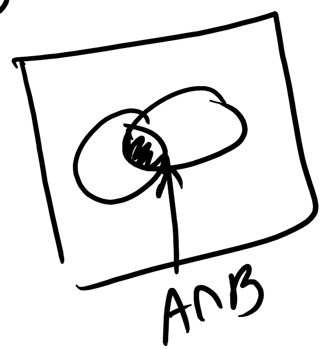
d) Recall "de Morgans" law:

$$\boxed{A' \cap B' = (A \cup B)'}$$ de Morgans law

$\Rightarrow P(A' \cap B') = P([A \cup B]') = \frac{1}{3}$

$(A \cap B') \cap (A \cap B) = \phi$





Notice that since $A \cup B$ is true when A is true, B true, or both true, it follows that to get $P(A \text{ or } B)$ we must subtract area corresponding to $A \cap B$. Thus,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B) - P(A \cap B)$$
$$= P(A) + P(B) - 2P(A \cap B)$$

Ex] Toss a coin 3 times. What is the probability of obtaining two heads?

Consider the sample space of outcomes:

$$\{HHH, \underbrace{HHT}_{e_2}, \underbrace{HTH}_{e_3}, \underbrace{THH}_{e_4}, HTT, THT, TTH, TTT\}$$

A: obtain two heads in 3 tosses

$P(A) = \frac{3}{8}$. Notice that $A = e_2 \cup e_3 \cup e_4$.

The probability that a composite event A will occur is the sum of probabilities of the simple events of which it is composed.

$$P(A) = P(e_2) + P(e_3) + P(e_4) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Notice that e_2, e_3, e_4 are mutually exclusive events.

Ex] What is the probability of getting at least 10 points in rolling two dice?

A_1: get 10 pts in two rolls

A_2: get 11 pts in two rolls

A_3: get 12 pts in two rolls

$$A_1 \cap A_2 \cap A_3 = \phi$$

C: get at least 10 points in rolling two dice

$C = A_1 \cup A_2 \cup A_3$ (union of 3 simple events)

$P(C) = P(A_1) + P(A_2) + P(A_3)$ since A_1, A_2, A_3 are mutually exclusive. They have no outcomes in common.

# of possible outcomes: $6 \times 6 = 36$ (two dice)

A_1: 6 and 4, 4 and 6, 5 and 5

$$P(A_1) = \frac{3}{36}$$

A_2: 6 and 5, 5 and 6

$$P(A_2) = \frac{2}{36}$$

A_3: 6 and 6 (get 12 pts in 2 rolls)

$$P(A_3) = \frac{1}{36}$$ (only one possibility)

$$\Rightarrow P(C) = P(A_1) + P(A_2) + P(A_3) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$$

$$= \frac{6}{36} = \frac{1}{6}$$

notice: key is that A_1, A_2, A_3 are mutually exclusive

For any two events (not necessarily mutually exclusive), one has:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Ex) Suppose one card is drawn at random from a deck of 52 cards.

A: get a "red ace"
B: get a "heart"

$P(A) = \frac{2}{52}$ (two kinds of red aces in 52 cards)

$P(B) = \frac{13}{52}$ (one of the four suits)

$P(A \cap B) = \frac{1}{52}$ (only a single ace of hearts card)

$$P(A \cup B) = P(A) + P(B) \boxed{- P(A \cap B)} = \frac{2}{52} + \frac{13}{52} - \frac{1}{52} = \frac{14}{52} = \frac{7}{26}$$

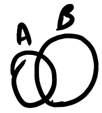
Notice: $A \cup B$ means get a red ace or a heart or an ace of hearts.

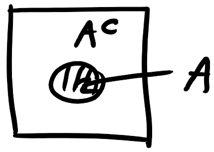
$A \cap B$ means get a red ace and a heart (in one draw, a single card).

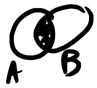
when we sum A and B, we count $A \cap B$ twice, so we need to subtract $P(A \cap B)$ from $P(A \cup B)$.

Set Relations

Independent of probability there are some fundamental relations between sets you should know.

A^c, A': complement of A 



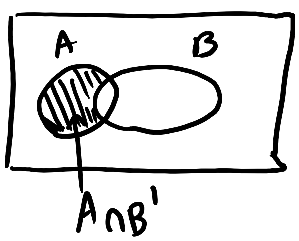
$A \cap B$ intersection  ; union $A \cup B$

$$A \cup B = (A \cap B') \cup B$$

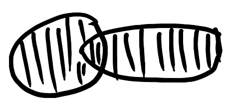
Consider x in $A \cup B$ (some element in $A \cup B$).

Then $x \in A$ or $x \in B$ or both.

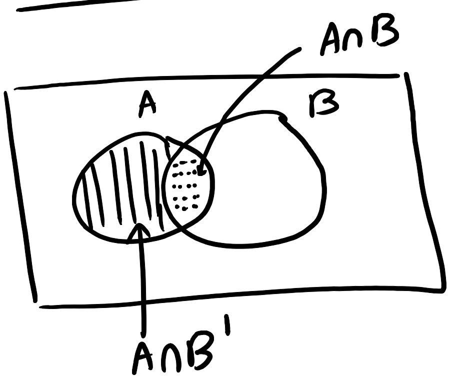
Can go through all cases and show that if an element x is in $A \cup B$ it is also in $(A \cap B') \cup B$ and vice versa.



Clear that $(A \cap B') \cup B$ corresponds to

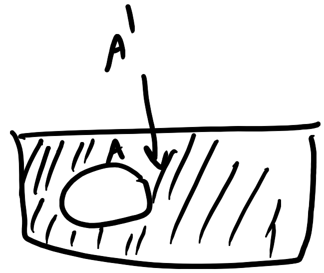
 : $A \cup B$

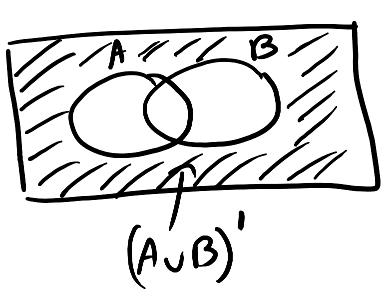
$$A = (A \cap B') \cup (A \cap B)$$



the combined region is set A

$$(A' \cap B') = (A \cup B)' \quad \text{(de Morgan's law)}$$



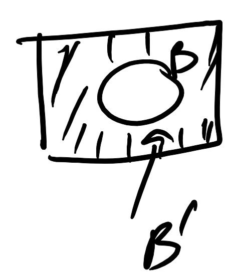


Application:

$P[(A' \cap B')] =$

$P[(A \cup B)'] = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$



Conditional probability

If $P(B) \neq 0$, then the conditional probability of A relative to B is:

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

(probability of A occuring given B)

Ex) A: randomly selected student comes from two parent home

B: student does poorly in school (avg $< D^+$)

$P(A) = 0.75$, $P(A \cap B) = 0.18$

What is the probability that a randomly selected student will be a low achiever given that he or she comes from a two-parent home.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.18}{0.75} \approx 0.24$$

does bad given that student comes from two parent home

Connection to event independence

Ex) Consider two flips of a fair coin

sample space: $\{HH, HT, TH, TT\}$

A: get head on first flip

B: get head on second flip

$P(A) = \frac{2}{4} = \frac{1}{2}$; $P(B) = \frac{1}{2}$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

← notice the order of B and A is important

$P(B \cap A) = P(A \cap B) = P(HH) = \frac{1}{4}$

$\Rightarrow P(B|A) = \frac{1/4}{1/2} = \frac{1}{2}$ (makes sense since A and B are independent!)

$P(B|A) = P(B) = \frac{1}{2}$

In this case, the probability of event B is the same regardless of whether event A has occured.

Ex) The probabilities that it will rain or snow in a given city on Christmas or New Years, or on both days are $P(C) = 0.60$, $P(N) = 0.60$, $P(C \cap N) = 0.42$. Check whether events N and C are independent.

$$P(N|C) = \frac{P(C \cap N)}{P(C)} = \frac{0.42}{0.60} = 0.70 \neq P(N)$$

Since $P(N|C) \neq P(N)$ the two events are dependent.

This makes sense; if there is a storm system it can influence weather for a week.

Ex) In rolling two fair dice, if the sum of two values is 7, what is the probability that one of the values is a 1.

A: one of the values is a 1

B: the sum of two rolls is 7

Review

Chapter 1 Intro, sampling techniques

Population vs sample. Biased vs unbiased.
parameter — statistic

Ex) voluntary response surveys likely to be biased.
Only people who feel strongly about it respond.

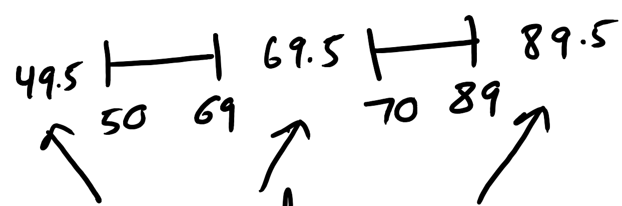
prepare and sample → analyze → conclude (stadistical vs practical significance)

Chapter 2

Frequency distributions and histograms

Ex1) (external) Notice that number of classes is either supplied or chosen by you. know lower/upper class limits and boundaries.

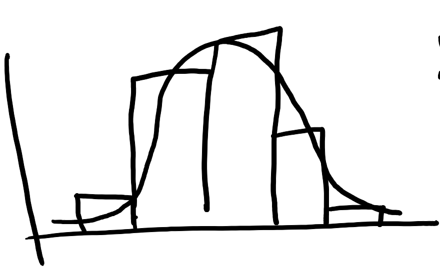
IQ score	frequency
50-69	2
70-89	33
90-109	35
110-129	7
130-149	1



class boundaries used to separate the classes.

Is it approximately normal?

yes, approx. bell shaped



Convert frequency distribution into a cumulative distribution.

Histograms

Can be drawn from a frequency distribution after class boundaries are defined.

The most importand use of histograms is to judge the distribution of the values in the data set.

Chapter 3 Measures of cendral tendency and variation.

Identification of outliers.

Compare mean and median (external).

mean $= \dfrac{\Sigma x}{n}$ median is middle of sorted data set.

mode is the measurement which occurs most frequently in the data set.

weighted mean : $\bar{x}_w = \dfrac{\Sigma(w \cdot x)}{\Sigma w}$ divide by sum of the weights

range $=$ max(x) $-$ min(x)

sample std dev : $s = \sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n-1}}$

population sample deviation : $\sigma = \sqrt{\dfrac{\Sigma(x-\mu)^2}{N}}$

s^2 is sample variance ; σ^2 is population variance

Ex) calculate the std deviation of the following sample

$\{2, 3, 3, 3, 4\}$

$$\bar{x} = \frac{2+3+3+3+4}{5} = 3$$

$$s = \sqrt{\frac{(2-3)^2 + (3-3)^2 + (3-3)^2 + (3-3)^2 + (4-3)^2}{5-1}} = \sqrt{\frac{2}{4}} = \sqrt{0.5}$$

"shortcut" formula for sample variance

$$s^2 = \frac{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}}{n-1}$$

$\bar{x}$ and s are estimators for μ, σ

x	x^2
2	4
3	9
3	9
3	9
4	16
$\Sigma x = 15$	$\Sigma x^2 = 47$

$$\Rightarrow s^2 = \frac{47 - \dfrac{(15)^2}{5}}{5-1} = \frac{47-45}{4} = \frac{2}{4} = 0.5$$

percentiles, median, quartiles

The median is the value of middle of sort(x) when n is odd, and mean of the two middle items when n is even.

Ex) $X = \{16, 10, 14, 13, 20, 11, 17\}$

$sort(x) = \{10, 11, 13, 14, 16, 17, 20\}$

$median(x) = 14$

what if we add 30 to the end?

$sort(X_{new}) = \{10, 11, 13, \underline{14, 16}, 17, 20, 30\}$

$median(X_{new}) = \frac{14+16}{2} = 15$

percentiles and quartiles

k^{th} percentile : at least $k\%$ of data equal or less than value. at least $100-k\%$ equal or greater than value.

<u>five number summary</u> first, obtain sort(x)

min(x), Q_1 (25th percentile), Q_2 (median), Q_3 (75th percentile), max(x)

converting k-th percentile to value: $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$

$sort(x) \rightarrow$ compute $L = \left(\frac{k}{100}\right) n$ where $n = \#$ of values

(a) if L is whole # $\rightarrow$ $val = \frac{[sort(x)]_L + [sort(x)]_{L+1}}{2}$

(b) if L is fraction $\rightarrow$ round up to I $\rightarrow$ $val = [sort(x)]_I$

Ex) consider the following temperature readings for June.

$X = \{90, 75, 86, 77, 85, 72, 78, 79, 94, 82, 74, 93\}$

$$\text{Sord}(x) = \{\underset{1}{72}, \underset{2}{74}, \underset{3}{75}, \underset{4}{77}, \underset{5}{78}, \underset{6}{79}, \underset{7}{82}, \underset{8}{85}, \underset{9}{86}, \underset{10}{90}, \underset{11}{93}, \underset{12}{94}\}$$

Q_1 is 25^{th} percentile ; $n=12$; $\min(x)=72$, $\max(x)=94$

$$L_{25} = \left(\frac{25}{100}\right)12 = \frac{1}{4}\cdot 12 = \frac{12}{4} = 3$$

$$Q_1 = \frac{\text{sort}(x)_3 + \text{sort}(x)_4}{2} = \frac{75+77}{2} = 76$$

$$Q_2 = \text{median} = \frac{79+82}{2} = 80.5 = 50^{th} \text{ percentile}$$

$$L_{75} = \left(\frac{75}{100}\right)12 = \left(\frac{15}{20}\right)12 = \left(\frac{3}{4}\right)12 = 9$$

$$Q_3 = \frac{\text{sord}(x)_9 + \text{sort}(x)_{10}}{2} = \frac{86+90}{2} = 88$$

$$IQR = Q_3 - Q_1 = 88 - 76 = 12$$

outliers are values greater than $Q_3 + 1.5IQR$
or less than $Q_1 - 1.5IQR$.

Recall also z-sores: $z = \dfrac{x - \bar{x}}{s}$

For approx. normally distributed data, can use z scores to characterize outliers ($z < -2$, $z > 2$). In that case, it is roughly equivalent to the above definitions.

Modified boxplots

any outliers would be marked with asterisks

Empirical Rule and chebyshev's theorem

ER: For normal data, 68% lies within 1 sd dev of mean, 95% lies within two, 99.7% lies within 3.

CT: For any data, at least $1-\frac{1}{k^2}$ of the data lie within k std deviations of the mean where k is any positive whole # greater than one.

$IQR = Q_3 - Q_1$; outliers $< Q_1 - 1.5IQR$, $> Q_3 + 1.5IQR$

z-scores: $z = \frac{x-\bar{x}}{s}$ ($|z| > 2 \rightarrow$ outliers)

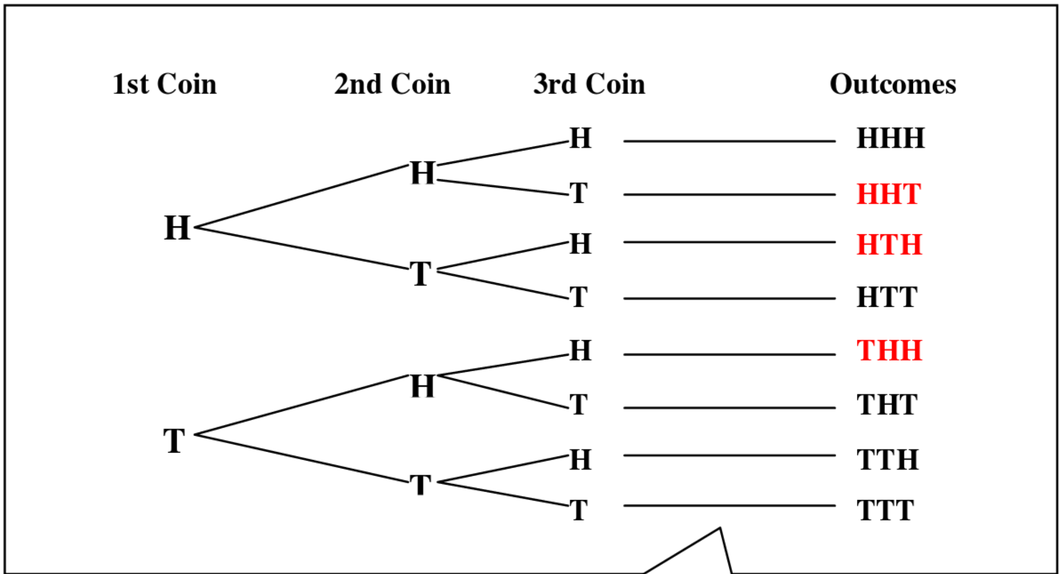
Two data sets $\{x, y\}$ where $x = \{x_1, \dots, x_n\}$ and $y = \{y_1, \dots, y_n\}$. Correlation coefficient, least squares regression. Modeling.

Probability. Always between 0 and 1.

Event: { The score was a tie }.

Simple event: { the score was 2 to 3 }.

Ex] If three coins are tossed what is the probability of getting exactly two heads?



Notice that there are multiple ways to get two heads. This is an example of an event which is not simple.

$$P(2 \text{ heads in } 3 \text{ tosses}) = \frac{3}{8}$$

mutually exclusive events A and B:

$A \cap B = \phi$ (empty set)

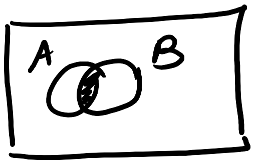
E.g.] A: you wear pants today
B: you wear shorts today

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$
in this case.

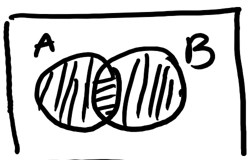
Independent events: occurrance of A does not influence B. E.g. A: heads on 1^{st} flip: B: heads on 2^{nd} flip.

Set relations

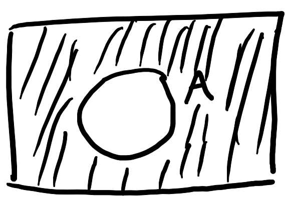
$A \cap B \Rightarrow$ A intersection B



$A \cup B \Rightarrow$ A union B (A true, B true or both true)

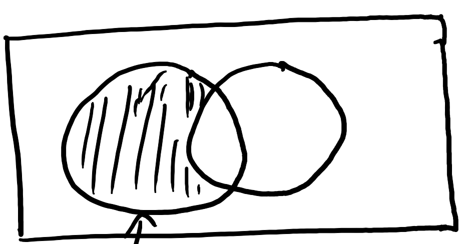


$A^C, A' \Rightarrow$ complement



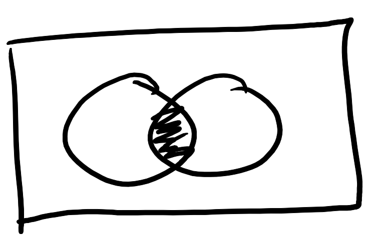
Important set relations:

$$A = A \cap B \cup (A \cap B')$$

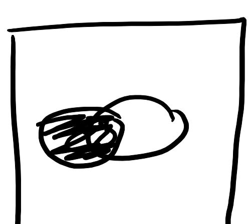


$A \cap B'$

$\cup$



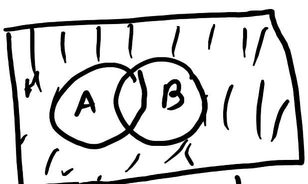
$=$



$$A \cup B = (A \cap B') \cup B$$

Also, De Morgan's Law:

$$A' \cap B' = (A \cup B)'$$



$A' \cap B'$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \underset{\text{exclusively}}{\text{ or }} B) = P(A) + P(B) - 2P(A \cap B)$$

Minus notation refers to A and not B:

$$A - B = A \cap B'$$